Math 3280A 2022-10-31

Review.

- Joint distribution of two r.v.'s
- Independence of two r.v.'s

Let X and Y be two r.v.'s. The joint cumulative distribution function (CDF) of X and Y is defined as

$$F(a,b) = P\{X \le a, Y \le b\}, \quad a, b \in \mathbb{R}.$$

Def. We say that X and Y are independent if

$$P\{X \in A, Y \in B\} = P\{X \in A\} P\{Y \in B\}$$

for all $A, B \subset \mathbb{R}$.

Theoretically, one can prove that

$$X \text{ and } Y \text{ are independent} \iff F(a,b) = F_X(a) F_Y(b) \quad \forall a, b \in \mathbb{R}.$$

- Moreover, when X, Y are both discrete,

$$X, Y \text{ are independent} \Leftrightarrow p(x,y) = p_X(x)\, p_Y(y), \ \forall x, y \in \mathbb{R}.$$

- When X and Y are jointly cts,

$$X \text{ and } Y \text{ are independent} \Leftrightarrow f(x,y) = f_X(x)\, f_Y(y) \quad \forall x, y \in \mathbb{R}.$$

§ 6.3 Sums of independent r.v.'s.

Question: Let X, Y be independent r.v.'s.

How to calculate the distribution of $X+Y$?

Suppose X, Y are jointly cts and independent. Then the density of $X+Y$ is given by

$$f_{X+Y}(a) = f_X * f_Y(a)$$

$$:= \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy$$

Example 1. Let X, Y be independent, both unif. dist. on $[0,1]$. Calculate the density of $X+Y$.

Solution: Let $a \in \mathbb{R}$. Then

$$f_{X+Y}(a) = f_X * f_Y(a)$$

$$= \int_{-\infty}^{\infty} f_X(a-y) f_Y(y)\, dy$$

$$= \int_0^1 f_X(a-y)\, dy \quad \left(\text{since } f_Y(y) = \begin{cases} 1 & \text{if } y \in [0,1] \\ 0 & \text{otherwise} \end{cases}\right)$$

$$\overset{\text{letting } z=a-y}{=} \int_{a-1}^{a} f_X(z)\, dz$$

If $0 < a \leq 1$,

$$\int_{a-1}^{a} f_X(z) dz = \int_0^a 1\, dz = a.$$

If $1 < a \leq 2$,

$$\int_{a-1}^{a} f_X(z) dz = \int_{a-1}^{1} 1\, dz = 2-a.$$

If $a > 2$ or $a < 0$,

$$\int_{a-1}^{a} f_X(z) dz = 0.$$

Hence

$$f_{X+Y}(a) = \begin{cases} a & \text{if } 0 < a \le 1 \\ 2-a & \text{if } 1 < a \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

Example 2. Let X, Y be independent normal r.v's with parameters $(0, 1)$ and $(0, \sigma^2)$. Find out the distribution of $X+Y$.

Solution: Recall that

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{y^2}{2\sigma^2}}, \quad y \in \mathbb{R}.$$

Hence for $a \in \mathbb{R}$,

$$f_X * f_Y(a) = \int_{-\infty}^{\infty} f_X(a-y)\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(a-y)^2}{2}} \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{y^2}{2\sigma^2}} dy$$

$$= \frac{1}{2\pi\sigma} \int_{-\infty}^{\infty} e^{-\frac{(a-y)^2}{2} - \frac{y^2}{2\sigma^2}} dy.$$

Notice that

$$\frac{(a-y)^2}{2} + \frac{y^2}{2\sigma^2} = \frac{\left(\sqrt{\sigma^2+1}\, y - \frac{a\sigma^2}{\sqrt{\sigma^2+1}}\right)^2}{2\sigma^2} + \frac{a^2}{2(\sigma^2+1)}$$

(verify it)

Hence

$$f_{X+Y}(a) = \frac{1}{2\pi\sigma} e^{-\frac{a^2}{2(\sigma^2+1)}} \int_{-\infty}^{\infty} e^{-\frac{\left(\sqrt{\sigma^2+1}\, y - \frac{a\sigma^2}{\sqrt{\sigma^2+1}}\right)^2}{2\sigma^2}} dy$$

$$\left(\text{letting } z = \frac{\sqrt{\sigma^2+1}\, y - \frac{a\sigma^2}{\sqrt{\sigma^2+1}}}{\sigma} \right)$$

$$= \frac{1}{2\pi\sigma} \cdot \frac{\sigma}{\sqrt{\sigma^2+1}} e^{-\frac{a^2}{2(\sigma^2+1)}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz$$

(Using the fact $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz = 1$)

$$= \frac{1}{\sqrt{2\pi}\sqrt{\sigma^2+1}} e^{-\frac{a^2}{2(\sigma^2+1)}}.$$

Hence $X+Y$ is a normal r.v. with parameters $(0, \sigma^2+1)$.

Remark: In general, if X, Y are independent, normal r.v.'s with parameters (μ_1, σ_1^2), and (μ_2, σ_2^2), then $X+Y$ has a normal distribution with parameters $(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$.