Math 3280A 2022-10-24

Review.

- Find the distribution of $g(X)$, where X is a cts rv with density f_X and $g: \mathbb{R} \to \mathbb{R}$.

Find

$$F_{g(X)}(y) := P\{ g(X) \le y \}$$

$$f_{g(X)}(y) = \frac{d}{dy} F_{g(X)}(y).$$

In the special case that g is strictly monotone,

$$f_{g(X)}(y) = \begin{cases} f_X(g^{-1}(y)) \cdot \left| \frac{d\, g^{-1}(y)}{dy} \right| & \text{if } y = g(x) \text{ for some } x \\ 0 & \text{otherwise.} \end{cases}$$

Chap 6. Joint distributed r.v.'s.

§ 6.1 Joint distribution.

Def. Let X, Y be two r.v.'s.

The joint cumulative distribution function of X, Y (joint CDF) is defined by

$$F(a,b) = P\{X \leq a, Y \leq b\}, \quad a, b \in \mathbb{R}.$$

From the above def, we see that F_X and F_Y can be obtained from $F(\cdot,\cdot)$.

$$
\begin{aligned}
F_X(a) &= P\{X \leq a\} \\
&= P\{X \leq a, Y < \infty\} \\
&= P\left(\lim_{b \to +\infty} \{X \leq a, Y \leq b\}\right) \\
&\overset{\text{using the continuity of } P}{=} \lim_{b \to +\infty} P\{X \leq a, Y \leq b\} \\
&= \lim_{b \to +\infty} F(a,b) \\
&=: F(a, +\infty)
\end{aligned}
$$

Similary, $F_Y(b) = \lim_{a \to +\infty} F(a,b) =: F(+\infty, b)$.

- Usually, F_X and F_Y are called the marginal distributions of X and Y.

Theoretically, all the joint statements about X and Y can be determined by the function $F(a,b)$.

Example 1: Suppose $F = F(a,b)$ is the joint CDF of X and Y.

Find $P\{X > a, Y > b\}$.

Solution:

$$P\{X>a, Y>b\} = 1 - P\left(\{X>a, Y>b\}^c\right)$$

$$= 1 - P\left(\{X \le a\} \cup \{Y \le b\}\right)$$

(using $P(E \cup F) = P(E) + P(F) - P(EF)$)

$$= 1 - P\{X \le a\} - P\{Y \le b\} + P\{X \le a, Y \le b\}$$

$$= 1 - F(a, \infty) - F(\infty, b) + F(a,b).$$

- Now we consider the case that both X and Y are discrete. In such case, we can define the joint prob. mass function of X and Y by (joint pmf)

$$p(x,y) = P\{X=x, Y=y\}.$$

Then

$$p_X(x) = P\{X=x\}$$

$$= \sum_y P\{X=x, Y=y\}$$

$$= \sum_y p(x,y).$$

similarly

$$p_Y(y) = \sum_x p(x,y).$$

In particular

$$F(a,b) = \sum_{\substack{(x,y)\\ x\le a,\, y\le b.}} p(x,y) .$$

- Def: We say two r.v's X and Y are jointly continuous if there exists $f: \mathbb{R}^2 \to [0,\infty)$ such that

$$P\{(X,Y)\in C\} = \iint_C f(x,y)\,dx\,dy,$$

for any "measurable" set $C \subset \mathbb{R}^2$.

("measurable" sets include, for instance, the countable union/intersections of rectangles $[a,b]\times[c,d]$)

- In particular,

$$P\{X \le a, Y \le b\} = P\{(X,Y)\in(-\infty,a)\times(-\infty,b)\}$$
$$= \int_{-\infty}^{b}\int_{-\infty}^{a} f(x,y)\,dx\,dy.$$

- The function f in the def is called the joint prob. density function of X and Y.

Example 2. Suppose X and Y have a joint density function

$$f(x,y) = \begin{cases} 12\, xy\,(1-x) & \text{if } 0<x<1,\ 0<y<1 \\ 0 & \text{otherwise.} \end{cases}$$

Find ① f_X, f_Y

② $E[X]$ and $E[Y]$.

Solution:

$$\underline{F_X(a)} = P\{X \le a\}$$

$$= P\{X \le a,\ Y < \infty\}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{a} f(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^a 12\, xy(1-x)\, dx\, dy$$

if $0 \le a \le 1$

$$= \int_0^1 12y \cdot \left(\frac{x^2}{2} - \frac{x^3}{3} \right) \Big|_0^a \, dy$$

$$= \int_0^1 12y \cdot \left(\frac{a^2}{2} - \frac{a^3}{3}\right) dy$$

$$= 6y^2 \left(\frac{a^2}{2} - \frac{a^3}{3}\right) \Big|_0^1$$

$$= 6\left(\frac{a^2}{2} - \frac{a^3}{3}\right).$$

Taking derivative gives

$$f_X(a) = \begin{cases} 6 \cdot \left(a - a^2\right). & 0 < a < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_0^1 x\, 6\left(x - x^2\right) dx$$

$$= 2x^3 - \frac{6}{4}x^4 \Big|_0^1$$

$$= 2 - \frac{6}{4} = \frac{1}{2}.$$

Example 2.

Suppose X and Y have a joint density function

$$f(x,y) = \begin{cases} e^{-(x+y)} & \text{if } 0<x<\infty,\ 0<y<\infty \\ 0 & \text{otherwise.} \end{cases}$$

Find the prob. density function of $\frac{X}{Y}$.

Solution:

Since $f(x,y)=0$ if $(x,y) \notin (0,\infty)\times(0,\infty)$, we may assume X, Y always take positive values. So is X/Y.

For $a>0$,

$$P\left\{\frac{X}{Y} \le a\right\} = P\{X \le aY\}$$

$$= \iint_{\{(x,y):\ x \le ay\}} f(x,y)\,dx\,dy$$

$$= \iint_{\{(x,y)\in(0,\infty)\times(0,\infty)\,:\,x\le ay\}} e^{-(x+y)}\,dx\,dy$$

$$= \int_0^\infty \int_0^{ay} e^{-(x+y)}\,dx\,dy$$

$$= \int_0^\infty e^{-y}\cdot\left(-e^{-x}\right)\Big|_0^{ay}\,dy$$

$$= \int_0^\infty e^{-y}\cdot\left(1-e^{-ay}\right)\,dy$$

$$= \int_0^\infty e^{-y}-e^{-(a+1)y}\,dy$$

$$= -e^{-y}+\frac{1}{1+a}e^{-(a+1)y}\Big|_0^\infty$$

$$= 1-\frac{1}{1+a}$$

Taking derivative gives

$$f_{\frac{X}{Y}}(a) = \begin{cases} \dfrac{1}{(1+a)^2} & a>0 \\ 0 & \text{otherwise}. \end{cases}$$

Prop. Suppose X and Y have a joint density function f. Then the marginal densities of X, Y are given by

$$f_X(a) = \int_{-\infty}^{\infty} f(a,y)\,dy, \qquad a \in \mathbb{R}$$

$$f_Y(b) = \int_{-\infty}^{\infty} f(x,b)\,dx, \qquad b \in \mathbb{R}.$$

Pf. Notice that

$$F_X(a) = P\{X \le a\}$$

$$= \int_{-\infty}^{a} \left(\int_{-\infty}^{\infty} f(x,y)\,dy \right) dx$$

Let $g(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

Then

$$F_X(a) = \int_{-\infty}^{a} g(x)\,dx$$

Taking derivatives gives

$$f_X(a) = g(a) = \int_{-\infty}^{\infty} f(a,y)\,dy$$

(assuming that g is cts at a)

Similarly

$$f_Y(b) = \int_{-\infty}^{\infty} f(x,b)\,dx.$$

(under some regularity assumptions on f)

§6.2. Independent rv's.

Recall that two events E and F are said to be independent if

$$P(EF) = P(E)\,P(F).$$

Def: Two r.v's X and Y are said to be independent if

(*) $$P\{X \in A, Y \in B\} = P\{X \in A\} P\{Y \in B\}$$

for all subsets A, B of $\mathbb{R}$.

In other word, the events $\{X \in A\}$ and $\{Y \in B\}$ are independent for all given A, B.

Clearly if X and Y are independent, then

(**) $$F(a,b) = F_X(a) F_Y(b), \quad \forall\, a, b \in \mathbb{R}$$

Reason:
$$\begin{aligned} F(a,b) &= P\{X \le a, Y \le b\} \\ &= P\{X \le a\} P\{Y \le b\} \\ &= F_X(a) F_Y(b). \end{aligned}$$

(*) $\Rightarrow$ (**) is clear.

But (**) $\Rightarrow$ (*) is also true.