

Review.

- Let X be a cts r.v. with density f .

For $g: \mathbb{R} \rightarrow \mathbb{R}$,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x) dx.$$

$$\begin{aligned} \text{Var}(X) &= E[(X-\mu)^2] = E[X^2] - \mu^2 \\ &= \int_{-\infty}^{\infty} x^2 f(x) dx - \mu^2, \end{aligned}$$

where $\mu = E[X]$.

- Uniform r.v. on $[a, b]$,

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

Cumulative distribution function for a cts r.v.

- Def. (Cumulative distribution function).

Let X be a cts r.v with density f .

We define

$$\begin{aligned} F_X(b) &= P\{X \leq b\} \\ &= \int_{-\infty}^b f(x) dx. \end{aligned}$$

Prop 1. If f is cts at b , then

$$F'_X(b) = f(b).$$

Pf. Notice that for $u \in \mathbb{R}$, $u \neq 0$,

$$\begin{aligned} \frac{F_X(b+u) - F_X(b)}{u} &= \frac{\int_{-\infty}^{b+u} f(x) dx - \int_{-\infty}^b f(x) dx}{u} \\ &= \frac{1}{u} \int_b^{b+u} f(x) dx \end{aligned}$$

since f is cts at b , so

$f(x) - f(b)$ is close to 0
when x is close to b ,

hence as $u \rightarrow 0$, $\frac{1}{u} \int_b^{b+u} f(x) dx \rightarrow f(b)$.
□

Example 2

Let X be a unif r.v over $(0, 1)$.

Find the density of X^2 .

Solution : $F_{X^2}(b) = P\{X^2 \leq b\}$

$$= \begin{cases} P\{-\sqrt{b} \leq X \leq \sqrt{b}\}, & \text{if } b > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } b > 1, \\ \sqrt{b} & \text{if } 0 < b \leq 1, \\ 0 & \text{if } b < 0. \end{cases}$$

Taking derivative we obtain

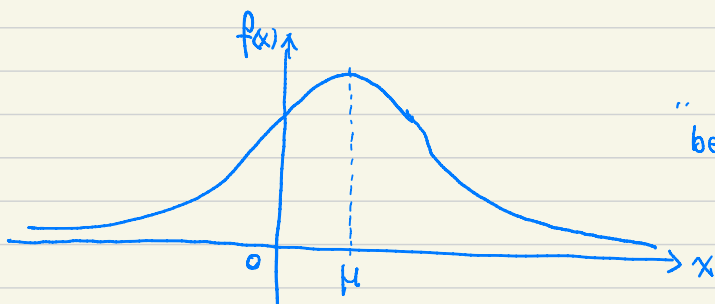
$$f_{X^2}(b) = \begin{cases} \frac{1}{2\sqrt{b}} & \text{if } b \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

□

§ 5.4 Normal r.v.

Def. Let $\mu \in \mathbb{R}$ and $\sigma > 0$. Say X is a normal r.v. with parameters μ and σ^2 if its pdf is given by

$$f(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$



"bell-shaped"

Fact 1

The above f is indeed a pdf.

To see this,

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \stackrel{\text{letting } y = \frac{x-\mu}{\sigma}}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy$$

We need to show $\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy = \sqrt{2\pi}$. Notice

$$(*) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{(x^2+y^2)}{2}} dx dy$$

$$\stackrel{\text{Letting } \begin{cases} x=r \cos \theta \\ y=r \sin \theta \end{cases}}{=} \int_0^{\infty} \int_0^{2\pi} e^{-\frac{r^2}{2}} r dr d\theta$$

$$= \int_0^{2\pi} \left(\int_0^{\infty} e^{-\frac{r^2}{2}} r dr \right) d\theta$$

$$= 2\pi \int_0^{\infty} e^{-\frac{r^2}{2}} r dr$$

$$= 2\pi \left(-e^{-\frac{r^2}{2}} \right) \Big|_0^{\infty}$$

$$= 2\pi.$$

Since $(*) = \left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy \right)^2$, we obtain the desired identity.

Fact 2

Let X be a normal r.v with parameters μ and σ^2 ,

then

$$Z := \frac{X - \mu}{\sigma}$$

is a normal r.v with parameters 0 and 1.

We call Z a standard normal r.v.

Pf. Calculate the cumulative distribution of Z ,

$$F_Z(a) = P\{Z \leq a\}$$

$$= P\left\{\frac{X - \mu}{\sigma} \leq a\right\}$$

$$= P\{X \leq a\sigma + \mu\}$$

$$= \int_{-\infty}^{a\sigma + \mu} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$\stackrel{\text{Letting } y = \frac{x-\mu}{\sigma}}{=} \int_{-\infty}^a \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

Taking derivative of $F_Z(a)$ gives

$$f_Z(a) = \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2}}$$

□

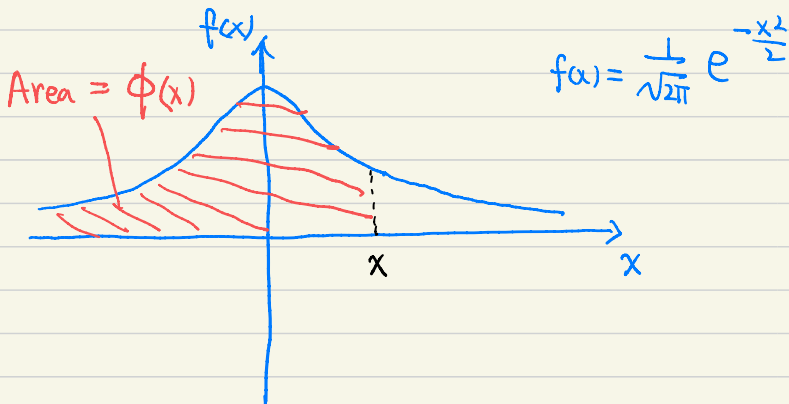
• Let

$$\Phi(x) = P\{Z \leq x\}$$

be the CDF of Z .

Then

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{y^2}{2}} dy$$



Fact 3

$$E[Z] = 0 \quad \text{and} \quad V(Z) = 1.$$

pf.

$$\begin{aligned} E[Z] &= \int_{-\infty}^{\infty} x \cdot f(x) dx \\ &= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Big|_{-\infty}^{\infty} \\ &= 0 \end{aligned}$$

$$\begin{aligned} E[Z^2] &= \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot x \cdot \left(-e^{-\frac{x^2}{2}}\right)' dx \end{aligned}$$

int by parts

$$= \frac{1}{\sqrt{2\pi}} \cdot x \cdot \left(-e^{-\frac{x^2}{2}}\right) \Big|_{-\infty}^{\infty}$$

$$+ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot (x)' dx$$

$$= 1$$

Hence

$$\begin{aligned}V(Z) &= E[Z^2] - (E[Z])^2 \\ &= 1 - 0^2 = 1. \quad \square\end{aligned}$$

Fact 4. Let $Y = aX + b$, where $a, b \in \mathbb{R}$ and X is a cts r.v.

Then ① $E[Y] = aE[X] + b$

② $V(Y) = a^2 V(X)$.

Pf. Let f be the pdf of X .

Then

$$\begin{aligned}E[Y] &= \int_{-\infty}^{\infty} (ax+b) f(x) dx \\ &= a \int_{-\infty}^{\infty} x f(x) dx + b \int_{-\infty}^{\infty} f(x) dx \\ &= a E[X] + b\end{aligned}$$

$$\begin{aligned} E[Y^2] &= \int_{-\infty}^{\infty} (ax+b)^2 f(x) dx \\ &= \int_{-\infty}^{\infty} (a^2x^2 + 2abx + b^2) f(x) dx \\ &= a^2 \int_{-\infty}^{\infty} x^2 f(x) dx + 2ab \int_{-\infty}^{\infty} x f(x) dx \\ &\quad + b^2 \int_{-\infty}^{\infty} f(x) dx \\ &= a^2 E[X^2] + 2ab E[X] + b^2 \end{aligned}$$

Hence

$$\begin{aligned} V(Y) &= E[Y^2] - (E[Y])^2 \\ &= a^2 E[X^2] + 2ab E[X] + b^2 \\ &\quad - (aE[X] + b)^2 \\ &= a^2 (E[X^2] - (E[X])^2) \\ &= a^2 V(X). \end{aligned}$$

□

Fact 5. Let X be a normal r.v. with parameters μ and σ^2 .

$$\text{Then } E[X] = \mu$$

$$V(X) = \sigma^2.$$

Pf. Let $Z = \frac{X - \mu}{\sigma}$. Then Z is a standard normal r.v. Hence $E[Z] = 0$, $V(Z) = 1$.

Now $X = \sigma Z + \mu$. By Lem 1,

$$E[X] = \sigma E[Z] + \mu = \sigma \cdot 0 + \mu = \mu.$$

$$V(X) = \sigma^2 V(Z) = \sigma^2 \cdot 1 = \sigma^2. \quad \square$$

The most important property of normal r.v is that it can be used to approximate a binomial r.v with parameters (n, p) when n is large.

Thm 3. (DeMoivre-Laplace Thm)

Let $0 < p < 1$. Let X_n be a binomial r.v with parameters (n, p) . Then for given $a, b \in \mathbb{R}$,

$$P \left\{ a < \frac{X_n - np}{\sqrt{np(1-p)}} < b \right\}$$

as $n \rightarrow \infty$

$$\longrightarrow P \left\{ a < Z < b \right\}$$

$$= \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= \Phi(b) - \Phi(a).$$

• In other word, for large n ,

$\frac{X_n - np}{\sqrt{np(1-p)}}$ has approximately the

same distribution as Z .

Example 4. Let X be a binomial r.v
with parameters $n=40$ and $p=\frac{1}{2}$.
Find $P\{X=20\}$.

Solution: We first find the precise value
of this prob.

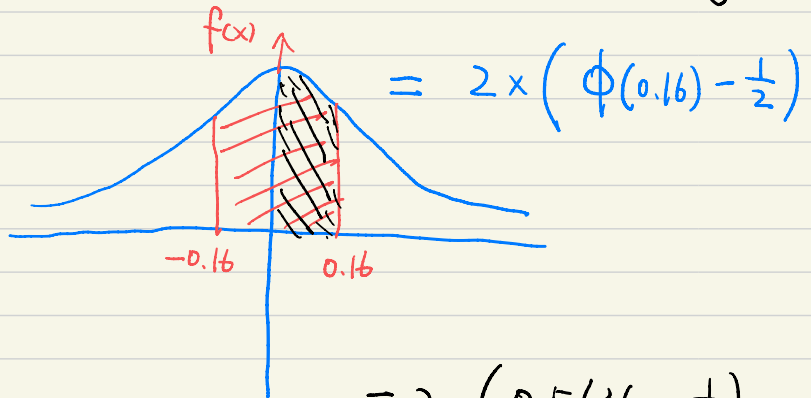
$$P\{X=20\} = \binom{40}{20} \cdot \left(\frac{1}{2}\right)^{40} \\ \approx .1254.$$

Next we estimate this prob. by using normal
distribution.

$$P\{X=20\} = P\{19.5 \leq X \leq 20.5\} \\ \text{(continuity correction)} \\ = P\left\{\frac{-0.5}{\sqrt{10}} \leq \frac{X-20}{\sqrt{10}} \leq \frac{0.5}{\sqrt{10}}\right\}$$

$$\approx P\left\{-\frac{0.5}{\sqrt{10}} \leq Z \leq \frac{0.5}{\sqrt{10}}\right\}$$

$$\approx P\{-0.16 \leq Z \leq 0.16\}$$



$$= 2 \times \left(0.5636 - \frac{1}{2}\right)$$

$$= 0.1272$$

Table 5.1 Area $\Phi(x)$ Under the Standard Normal Curve to the Left of X .

X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
<u>.1</u>	.5398	.5438	.5478	.5517	.5557	.5596	<u>.5636</u>	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\phi(0.16)$

The values of $\Phi(x)$ for nonnegative x are given in Table 5.1. For negative values of x , $\Phi(x)$ can be obtained from the relationship

$$\Phi(-x) = 1 - \Phi(x) \quad -\infty < x < \infty \quad (4.1)$$

The proof of Equation (4.1), which follows from the symmetry of the standard normal density, is left as an exercise. This equation states that if Z is a standard normal random variable, then

$$P\{Z \leq -x\} = P\{Z > x\} \quad -\infty < x < \infty$$