Math 3280A 22-10-10

Review

- Continuous r.v. X : $\exists$ a non-negative function f on $\mathbb{R}$ such that

$$P\{X \in B\} = \int_B f(x)\,dx, \quad \forall \text{ "measurable" set } B \subset (-\infty, \infty),$$

where "measurable" sets include all intervals and countable unions / intersections of intervals.

In particular, $P\{a \leq X \leq b\} = \int_a^b f(x)\,dx$.

and $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

We call f the pdf of X.

- $$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx.$$

Prop: Let X be a cts r.v. with density f.

Let $g: \mathbb{R} \to \mathbb{R}$. Then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx.$$

We prove the above prop only in the case that $g \geq 0$.

Lem. Let Y be a non-negative cts r.v. Then

$$E[Y] = \int_0^{\infty} P\{Y > y\}\, dy.$$

Proof of the proposition: By the above lem,

$$E[g(X)] = \int_0^{\infty} P\{g(X) > y\}\, dy.$$

Write $B := \{x \in \mathbb{R} : g(x) > y\}$. Then

$$P\{g(X) > y\} = P\{X \in B\} = \int_B f(x)\, dx = \int_{\{x : g(x) > y\}} f(x)\, dx.$$

Hence

$$\begin{aligned} E[g(X)] &= \int_0^{\infty} \int_{\{x : g(x) > y\}} f(x)\, dx \; dy \\ &= \int_0^{\infty} \left(\int_{-\infty}^{\infty} \mathbb{1}_{\{g(x) > y\}} f(x)\, dx \right) dy \end{aligned}$$

(where $\mathbb{1}_{\{g(x)>y\}} := h(x,y)$

$$= \begin{cases} 1 & \text{if } g(x)>y \\ 0 & \text{otherwise} \end{cases}$$)

$$\overset{(\text{Fubini})}{=} \int_{-\infty}^{\infty} \left(\int_0^{\infty} \mathbb{1}_{\{g(x)>y\}} f(x)\, dy \right) dx$$

$$= \int_{-\infty}^{\infty} f(x) \left(\int_0^{\infty} \mathbb{1}_{\{g(x)>y\}} dy \right) dx$$

$$= \int_{-\infty}^{\infty} f(x) \left[\int_0^{g(x)} 1\, dy + \int_{g(x)}^{\infty} 0\, dy \right] dx$$

$$= \int_{-\infty}^{\infty} f(x)\, g(x)\, dx.$$

$\boxtimes$

Def: (Variance) Let X be a cts r.v. with density f.

$$V(X) = E[(X-\mu)^2], \quad \text{where } \mu = E[X].$$

- $$V(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\, dx - \mu^2.$$

Pf. $V(X) = \int_{-\infty}^{\infty} (x^2 - 2x\mu + \mu^2) f(x)\, dx$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - 2\mu \int_{-\infty}^{\infty} x f(x)\,dx + \mu^2 \int_{-\infty}^{\infty} f(x)\,dx$$

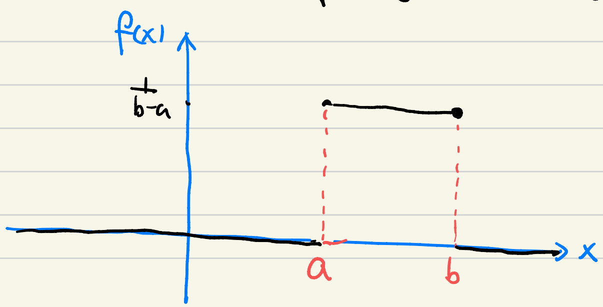
$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - 2\mu\cdot\mu + \mu^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2. \quad \square$$

§ 5.3. Uniform distributions.

Def. A cts X is said to be uniformly distributed on $[a,b]$ if it has a density

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if} \quad x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$



We call X is a unif. r.v on $[a,b]$, or say X has a unif distribution on $[a,b]$.

<u>Example 1</u>: Calculate $E[X]$ and $V(X)$ for a unif r.v. X on $[a,b]$.

<u>Solution</u>:

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_a^b x \frac{1}{b-a}\,dx$$

$$= \frac{\frac{1}{2}x^2}{b-a}\Big|_a^b = \frac{\frac{1}{2}(b^2-a^2)}{b-a}$$

$$= \frac{a+b}{2}.$$

$$\int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_a^b x^2 \frac{1}{b-a}\,dx$$

$$= \frac{1}{3}\frac{b^3-a^3}{(b-a)} = \frac{a^2+ab+b^2}{3}$$

Hence

$$V(X) = \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4}$$

$$= \frac{(a-b)^2}{12}.$$

- Def. (Cumulative distribution function).

Let X be a cts r.v with density f.

We define

$$F_X(b) = P\{X \leq b\}$$

$$= \int_{-\infty}^{b} f(x)\,dx.$$

Prop. If f is cts at b, then

$$F_X'(b) = f(b).$$

Pf. Notice that for $u \in \mathbb{R}$, $u \neq 0$,

$$\frac{F_X(b+u) - F_X(b)}{u} = \frac{\int_{-\infty}^{b+u} f(x)\,dx - \int_{-\infty}^{b} f(x)\,dx}{u}$$

$$= \frac{1}{u}\int_b^{b+u} f(x)\,dx$$

Since f is cts at b, so

$f(x) - f(b)$ is close to 0

when x is close to b,

hence as $u \to 0$, $\frac{1}{u}\int_b^{b+u} f(x)\,dx \to f(b)$.

$\square$.

Let X be a unif r.v over $(0,1)$.

Find the density of X^2.

Solution : $F_{X^2}(b) = P\{X^2 \le b\}$

$$= \begin{cases} P\{-\sqrt{b} \le X \le \sqrt{b}\}, & \text{if } b > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1 & \text{if } b > 1, \\ \sqrt{b} & \text{if } 0 < b \le 1, \\ 0 & \text{if } b < 0. \end{cases}$$

Taking derivative we obtain

$$f_{X^2}(b) = \begin{cases} \frac{1}{2\sqrt{b}} & \text{if } b \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

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