Math 3280A 22-10-06

Review

- Poisson r.v with parameter λ,

$$P\{X=k\} = e^{-\lambda}\cdot\frac{\lambda^k}{k!}, \quad k=0,1,2,\cdots$$

It can be used to approximate binomial r.v.'s.

- $E[X_1+\cdots+X_n] = E[X_1]+E[X_2]+\cdots+E[X_n]$.

§ 4.9. Cumulative distribution function.

Def. Let X be a discrete r.v. Define

$$F_X(b) = P\{X\leq b\}, \quad b\in\mathbb{R}.$$

We call F_X the cumulative distribution function (CDF) of X. We also write $F(b)=F_X(b)$.

Prop 1. (1) F is non-decreasing, that is

$$F(a) \leq F(b) \quad \text{if} \quad a < b.$$

(2) $\lim_{b \to +\infty} F(b) = 1.$

(3) $\lim_{b \to -\infty} F(b) = 0.$

(4) F is right continuous, i.e.

$$\lim_{b_n \downarrow b} F(b_n) = F(b).$$

(i.e. b_n tends to b from the RHS of b)

The prop. is based on the continuity property of probability.

Recall that if $E_n \nearrow E$, then $\lim P(E_n) = P(E)$

(i.e $E_{n+1} \supset E_n$, $E = \bigcup_{n=1}^{\infty} E_n$)

if $E_n \searrow E$ ($E_{n+1} \subset E_n$, $E = \bigcap_{n=1}^{\infty} E_n$)

then $P(E_n) \to P(E)$ as $n \to \infty$.

pf of Prop 1.

(1) Since if $a < b$,

then $\{X \leq a\} \subset \{X \leq b\}$

So $F(a) \leq F(b)$.

(2) If $b_n \nearrow \infty$,

then $\{X \leq b_n\} \nearrow \{X < \infty\} = S$

So $F(b_n) \to 1$ (by the continuity property of probability)

(3) If $b_n \searrow -\infty$, then

$$\{X \leq b_n\} \searrow \{X = -\infty\} = \emptyset$$

So $F(b_n) \to 0$.

(4) If $b_n \searrow b$, then

$\{X \leq b_n\} \searrow \{X \leq b\}$.

So $F(b_n) \to F(b)$. Thus F is right cts.

Remark: • In general, F is not left continuous.

• In the discrete case, one has

$$P\{X=b\} = F(b) - \lim_{b_n \uparrow b} F(b_n)$$
$$= F(b) - F(b-)$$

Reason: If $b_n \uparrow b$, then

$$\{X \le b_n\} \uparrow \{X < b\}.$$

So $P\{X=b\} = P\{X \le b\} - P\{X<b\}$.

Chap 5. Continuous r.v.'s.

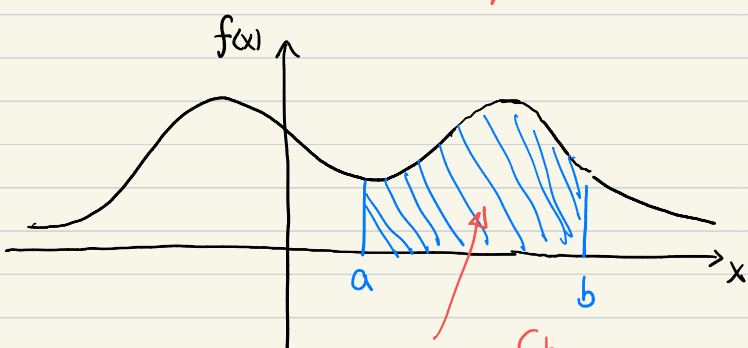
§ 5.1 Introduction.

Def. X is called a cont. r.v. (or absolute cont. r.v.) if there is a non-negative function defined on $(-\infty, \infty)$ such that

$$P\{X \in B\} = \int_B f(x)\,dx$$

for all "measurable" sets $B \subset (-\infty, \infty)$.

Remark: The measurable sets include all intervals, and the countable unions / intersections of intervals.



$$\text{Area} = \int_a^b f(x)\,dx$$

$$P\{a \leq X \leq b\} = \int_a^b f(x)\,dx$$

$$= \text{Area of the shaded region.}$$

- We call f the prob. density function (pdf) of X.

Example 2. Let X be a cts r.v. with pdf

$$f(x) = \begin{cases} C(4x-2x^2) & \text{if } x \in (0,2) \\ 0 & \text{otherwise.} \end{cases}$$

(1) Find the value of C

(2) Find $P\{X \geq 1\}$.

- $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1.$

Since $P\{X \in (-\infty, \infty)\} = P\{S\} = 1$

Solution:

$$1 = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_0^2 c\left(4x - 2x^2\right) dx$$

$$= c \cdot \left(2x^2 - \frac{2}{3}x^3\right)\Big|_0^2$$

$$= c \cdot \left(8 - \frac{2}{3} \times 8\right) = \frac{8}{3}c.$$

Hence $c = \frac{3}{8}$.

$$P\{X \geq 1\} = \int_1^{\infty} f(x)\,dx$$

$$= \int_1^2 \frac{3}{8}\left(4x - 2x^2\right) dx$$

$$= \frac{3}{8}\left(2x^2 - \frac{2}{3}x^3\right)\Big|_1^2$$

$$= \frac{1}{2}.$$

Exer. 3. Suppose X has a pdf

$$f(x) = \begin{cases} \lambda e^{-\frac{x}{100}} & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

(1) Find the value of λ.

(2) Find $P\{X \leq 100\}$.

• Solution:

$$1 = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_{0}^{\infty} \lambda e^{-\frac{x}{100}}\,dx$$

$$= \lambda \cdot \left(-100\, e^{-\frac{x}{100}}\right)\Big|_{0}^{\infty}$$

$$= \lambda \cdot 100$$

Hence $\lambda = \frac{1}{100}$.

$$P\{X \le 100\} = \int_{-\infty}^{100} f(x)\,dx$$

$$= \int_0^{100} \frac{1}{100} e^{-\frac{x}{100}}\,dx$$

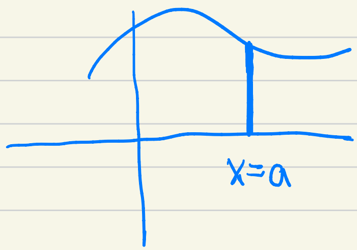
$$= -e^{-\frac{x}{100}} \Big|_0^{100}$$

$$= 1 - e^{-1}. \qquad \blacksquare$$

Remark: In the continuous case,

$$P\{X = a\} = 0 \quad \text{for any } a \in \mathbb{R}.$$

Reason:



Area of the line $x=a$ below $y=f(x)$ is 0.

Also

$$P\{X=a\} = \int_a^a f(x)\,dx = 0.$$

Hence

$$P\{a\le X\le b\} = P\{a< X\le b\}$$

$$= P\{a< X< b\}.$$

§ 5.2 Expectation of a cts. r.v.

Def. Let X be a cts r.v. with pdf f.

Then we define

$$E[X] = \int_{-\infty}^{\infty} x f(x)\,dx.$$

Remark: Recall that in the discrete case,

$$E[X] = \sum x P\{X = x\}.$$

In the continuous case, we make a partition of $(-\infty, \infty)$ by $(x_n)_{n=-\infty}^{\infty}$ such that

$$x_{n+1} - x_n = \Delta x$$

$$\sum_n x_n \cdot P\{x_n < X \leq x_{n+1}\}$$

$$= \sum_n x_n \int_{x_n}^{x_n+\Delta x} f(x)\, dx$$

$$\approx \sum_n x_n f(x_n) \cdot \Delta x \longrightarrow \int_{-\infty}^{\infty} x f(x)\, dx$$

as $\Delta x \to 0$.

Example 4. X is said to be uniformly distributed on $[0,1]$ if it has the following density

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Find $E[X]$.

Solution:

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_0^1 x \cdot 1\, dx$$

$$= \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}.$$

Prop 5. Let Y be a non-negative cts r.v.

Then $$E[Y] = \int_0^\infty P\{Y > y\}\, dy.$$

To prove this result, we use the following

$$\int_a^b \left(\int_c^d f(x,y)\, dx \right) dy = \int_c^d \left(\int_a^b f(x,y)\, dy \right) dx$$

when f is non-negative. (Fubini Thm).

Pf of Prop 2. Let f be the density of Y.

Then

$$\int_0^\infty P\{Y > y\}\, dy$$

$$= \int_0^\infty \left(\int_y^\infty f(x)\, dx \right) dy$$

$$= \int_0^\infty \left(\int_0^\infty g(x,y) \quad f(x)\, dx \right) dy$$

(where g is defined by

$$g(x,y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{otherwise} \end{cases}$$
)

By Fubini

$$= \int_0^\infty \left(\int_0^\infty g(x,y) \quad f(x)\, dy \right) dx$$

$$= \int_0^\infty f(x) \left(\int_0^\infty g(x,y) \quad dy \right) dx$$

$$= \int_0^\infty f(x) \cdot x \, dx$$

$\blacksquare$

(Notice $g(x,y) = \begin{cases} 1 & \text{if } \quad y < x \\ 0 & \text{otherwise} \end{cases}$.

Hence $\int_0^\infty g(x,y)\, dy = \int_0^x g(x,y)\,dy + \int_x^\infty g(x,y)\, dy$

$$= \int_0^x 1\, dy + \int_x^\infty 0\, dy$$

$$= x. \Big)$$