Math 3280A 2022-11-17

Review

- Conditional expectation $E[X|Y=y]$
- Calculate expectation by conditioning

$$E[X] = E[E[X|Y]]$$

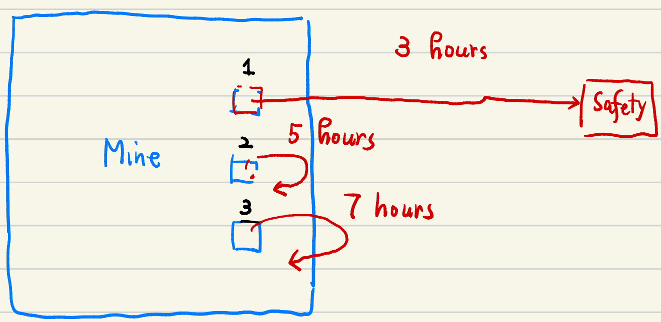
If Y is a discrete r.v., then we have

$$E[X] = \sum_y E[X|Y=y]\, P\{Y=y\}.$$

Below we give an example.

Example 1.

A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third door leads to a tunnel that will return him to the mine after 7 hours. If we assume that the miner is at all times equally likely to choose any one of the doors, what is the expected length of time until he reaches safety?



Solution: Let X denote the length of time (in hours) until the miner reaches safety.

Let Y denote the door that he choose in the first time.

By Prop 2,

$$E[X] = E[\,E[X|Y]\,]$$

$$= E[X|Y=1]\cdot P\{Y=1\}$$

$$+ E[X|Y=2]\cdot P\{Y=2\}$$

$$+ E[X|Y=3] \cdot P\{Y=3\}$$

$$= \frac{1}{3}\left(E[X|Y=1] + E[X|Y=2] + E[X|Y=3] \right)$$

$$= \frac{1}{3}\left(3 + (5 + E[X]) + (7 + E[X]) \right).$$

Solving this equation, we obtain

$$E[X] = 3 + 5 + 7 = 15 \quad \text{(hours)}$$

§ 7.7 Moment generating functions.

Def. Let X be a r.v. and $t \in \mathbb{R}$. Define

$$M_X(t) = E[e^{tX}].$$

For convenience, we also write $M(t) = M_X(t)$ and call it the moment generating function of X.

Remark:

① $$e^{tX} = \sum_{n=0}^{\infty} \frac{1}{n!} t^n \cdot X^n.$$

Hence

(1) $$M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} E[X^n]$$

$\to$ n-th moment of X.

② If $M_X(t)$ exists and is finite for all $-t_0 < t < t_0$ for some $t_0 > 0$,

then

(2) $$E[X^n] = M_X^{(n)}(0), \quad n = 1, 2, \dots$$

Example 2. Let X be a binomial r.v. with parameters (n, p).

$$M_X(t) = E[e^{tX}] = \sum_{k=0}^{n} e^{tk} P\{X = k\}$$

$$= \sum_{k=0}^{n} e^{tk} \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k} (e^t p)^k (1-p)^{n-k}$$

$$\overset{\text{By Binomial Thm}}{=} \left(e^t p + (1-p)\right)^n.$$

Example 3. Let X be a Poisson r.v. with parameter λ.

$$M_X(t) = E[e^{tX}] = \sum_{k=0}^{\infty} e^{tk} \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= \sum_{k=0}^{\infty} e^{-\lambda} \cdot \frac{(e^t \cdot \lambda)^k}{k!}$$

$$= e^{-\lambda} \cdot e^{(e^t \lambda)}$$

$$= e^{\lambda(e^t - 1)}.$$

Example 4. Let Z be a standard normal r.v.

$$M_Z(t) = E[e^{tZ}]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tz} e^{-\frac{z^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{t^2}{2}} \cdot e^{-\frac{(z-t)^2}{2}} dz$$

$$\overset{\text{Letting } x = z-t}{=\!=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{t^2}{2}} e^{\frac{-x^2}{2}} dx$$

$$= e^{t^2/2}$$

Example 5. Let X be a normal r.v. with mean μ and variance σ^2.

Notice that $Z := \frac{X-\mu}{\sigma}$ is a standard normal r.v.

Hence $X = \mu + \sigma Z$.

$$
\begin{aligned}
M_X(t) &= E[e^{t(\mu+\sigma Z)}] \\
&= E[e^{t\mu} \cdot e^{t\sigma Z}] \\
&= e^{t\mu} E[e^{t\sigma Z}] \\
&= e^{t\mu} \cdot M_Z(t\sigma) \\
&= e^{t\mu} \cdot e^{\frac{t^2\sigma^2}{2}} = e^{\frac{\sigma^2 t^2}{2} + \mu \cdot t}
\end{aligned}
$$

Thm 6. Let X, Y be two r.v.'s.

If $\exists\ t_0 > 0$ such that

$$M_X(t) = M_Y(t) \quad \text{for} \quad t \in (-t_0, t_0)$$

and both are finite.

Then X and Y have the same distribution

(i.e $F_X = F_Y$)

Due to this result, we say that the moment generating function determines the distribution.

Prop 7. If X and Y are independent then $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$.

Pf. $M_{X+Y}(t) = E[e^{tX+tY}]$

$= E[e^{tX} \cdot e^{tY}]$

$= E[e^{tX}] \cdot E[e^{tY}]$ (here we use the independency of X and Y)

$= M_X(t) \cdot M_Y(t).$

$\boxed{/\!/}$

Example 8. Let X, Y be independent Poisson r.v.'s with parameters λ_1, λ_2, respectively.

Show that $X+Y$ is a Poisson r.v. with parameter $\lambda_1+\lambda_2$.

Pf. Notice that by Example 3,

$$M_X(t) = e^{\lambda_1(e^t-1)}, \quad M_Y(t) = e^{\lambda_2(e^t-1)}.$$

Since X and Y are independent,

$$M_{X+Y}(t) = M_X(t)\, M_Y(t) = e^{(\lambda_1+\lambda_2)(e^t-1)}.$$

By Example 3 and Thm 6, $X+Y$ has the Poisson distribution with parameter $\lambda_1+\lambda_2$. $\boxed{/\!/}$