

Review

$$\bullet E[X_1 + \dots + X_n] = \sum_{i=1}^n E[X_i]$$

(linearity of expectation)

- For two r.v.'s X and Y ,

$$\text{Cov}(X, Y) = E[(X - E[X]) \cdot (Y - E[Y])]$$

In particular, $\text{Cov}(X, X) = \text{Var}(X)$.

Remark: $\text{Cov}(X, Y) = E[XY] - E[X] \cdot E[Y]$,

Pf. Write $\mu = E[X]$, $\nu = E[Y]$.

$$\text{Then } \text{Cov}(X, Y) = E[(X - \mu)(Y - \nu)]$$

$$= E[XY - \mu Y - \nu X + \mu \nu]$$

$$= E[XY] - \mu E[Y] - \nu E[X] + \mu \nu$$

$$= E[XY] - \mu \nu. \quad \square$$

• If X and Y are independent, then

$$\text{Cov}(X, Y) = 0.$$

Prop 1. (1) $\text{Cov}(X, Y) = \text{Cov}(Y, X).$

(2) $\text{Cov}(X, X) = \text{Var}(X).$

(3) $\text{Cov}(aX, Y) = a \text{Cov}(X, Y), \quad a \in \mathbb{R}.$

(4)
$$\text{Cov}\left(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j\right) \\ = \sum_{i=1}^n \sum_{j=1}^m \text{Cov}(X_i, Y_j).$$

(1), (3), (4) imply that $\text{Cov}(\cdot, \cdot)$ is bi-linear.

Pf. Let us prove (4) only. Write $\mu_i = E[X_i],$
 $\nu_j = E[Y_j].$

Then by definition,

$$\text{Cov}\left(\sum_{i=1}^n X_i, \sum_{j=1}^m Y_j\right) = E\left[\left(\sum_{i=1}^n X_i - \sum_{i=1}^n \mu_i\right) \cdot \left(\sum_{j=1}^m Y_j - \sum_{j=1}^m \nu_j\right)\right]$$

$$= E \left[\sum_{i=1}^n (X_i - \mu_i) \sum_{j=1}^m (Y_j - \nu_j) \right]$$

$$= E \left[\sum_{i=1}^n \sum_{j=1}^m (X_i - \mu_i)(Y_j - \nu_j) \right]$$

By the linearity of E

$$= \sum_{i=1}^n \sum_{j=1}^m E \left[(X_i - \mu_i)(Y_j - \nu_j) \right]$$

$$= \sum_{i=1}^n \sum_{j=1}^m \text{Cov}(X_i, Y_j).$$

□

Corollary 2.
$$\text{Var} \left(\sum_{i=1}^n X_i \right) = \sum_{i=1}^n \text{Var}(X_i) + 2 \sum_{1 \leq i < j \leq n} \text{Cov}(X_i, X_j).$$

Moreover if X_1, \dots, X_n are pairwise independent,

then

$$\text{Var} \left(\sum_{i=1}^n X_i \right) = \sum_{i=1}^n \text{Var}(X_i).$$

Example 1. Let X_1, \dots, X_n be independent and identical distributed, having expected value μ and variance σ^2 .

$$\text{Let } \bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \quad (\text{sample mean})$$

$$S^2 = \sum_{i=1}^n \frac{(X_i - \bar{X})^2}{n-1} \quad (\text{sample variance})$$

Find out ① $\text{Var}(\bar{X})$.

② $E[S^2]$.

Solution: $\text{Var}(\bar{X}) = \text{Var}\left(\frac{X_1 + \dots + X_n}{n}\right)$

$$= \frac{1}{n^2} \text{Var}(X_1 + \dots + X_n)$$

(notice X_1, \dots, X_n
are independent)

$$= \frac{1}{n^2} \sum_{i=1}^n \text{Var}(X_i)$$

$$= \frac{n \cdot \sigma^2}{n^2} = \frac{\sigma^2}{n}$$

Recall that
$$S^2 = \frac{\sum_{i=1}^n (X_i - \bar{X})^2}{n-1}$$

So

$$\begin{aligned}(n-1)S^2 &= \sum_{i=1}^n (X_i - \bar{X})^2 \\ &= \sum_{i=1}^n ((X_i - \mu) - (\bar{X} - \mu))^2 \\ &= \sum_{i=1}^n \left[(X_i - \mu)^2 - 2(X_i - \mu)(\bar{X} - \mu) \right. \\ &\quad \left. + (\bar{X} - \mu)^2 \right] \\ &= \sum_{i=1}^n (X_i - \mu)^2 - 2(\bar{X} - \mu) \cdot \sum_{i=1}^n (X_i - \mu) \\ &\quad + n \cdot (\bar{X} - \mu)^2 \\ &= \sum_{i=1}^n (X_i - \mu)^2 - 2n(\bar{X} - \mu)^2 + n(\bar{X} - \mu)^2 \\ &= \sum_{i=1}^n (X_i - \mu)^2 - n(\bar{X} - \mu)^2\end{aligned}$$

Hence
$$\begin{aligned}E[(n-1)S^2] &= \sum_{i=1}^n E[(X_i - \mu)^2] \\ &\quad - n E[(\bar{X} - \mu)^2] \\ &= n \cdot \sigma^2 - n \cdot \frac{\sigma^2}{n}\end{aligned}$$

$$= (n-1)\sigma^2.$$

Thus $E[S^2] = \sigma^2$. \square

Example 2. Define \bar{X} as in Example 1.

Show that $\text{Cov}(X_i - \bar{X}, \bar{X}) = 0$

Pf. $\text{Cov}(X_i - \bar{X}, \bar{X})$

$$= \text{Cov}(X_i, \bar{X}) - \text{Cov}(\bar{X}, \bar{X})$$

$$= \text{Cov}\left(X_i, \frac{X_1 + \dots + X_n}{n}\right) - \frac{\sigma^2}{n}$$

$$= \frac{1}{n} \sum_{j=1}^n \text{Cov}(X_i, X_j) - \frac{\sigma^2}{n}$$

$$= \frac{1}{n} \left[\text{Cov}(X_i, X_i) + \sum_{j \neq i} \text{Cov}(X_i, X_j) \right]$$

$$= \frac{1}{n} \text{Var}(X_i) - \frac{\sigma^2}{n} = \frac{\sigma^2}{n} - \frac{\sigma^2}{n} = 0.$$