Math 3280 22-09-22

Review.

- Conditional probability

$$P(E|F) = P(EF)/P(F), \quad \text{if } P(F) > 0.$$

(conditional prob. of E given F)

- $$P(E_1 E_2 \cdots E_n) = P(E_1)\cdot P(E_2|E_1)\cdot P(E_3|E_1E_2)\cdots P(E_n|E_1\cdots E_{n-1})$$

(Multiplicative rule).

- Suppose $F_1, F_2, \cdots, F_n$ are mutually exclusive, and exhausitive (i.e. $\bigcup_{i=1}^{n} F_i = S$).

Then

- $$P(E) = \sum_{i=1}^{n} P(F_i)\cdot P(E|F_i).$$

(law of total probability)

- $$P(F_i|E) = \frac{P(F_i)\,P(E|F_i)}{\sum_{k=1}^{n} P(F_k)\,P(E|F_k)}.$$

(Bayes' formula).

Remark: Suppose $F \subset S$ is an event in a sample space with $P(F) > 0$.

Then $P(\cdot | F)$ is a probability on S.

(1) $P(S|F) = 1$.

(2) $0 \leq P(E|F) \leq 1$.

(3) $P\left(\left(\bigcup_{n=1}^{\infty} E_n\right) \Big| F\right) = \sum_{n=1}^{\infty} P(E_n|F)$, if $E_1, E_2, \cdots$ are mutually exclusive.

(1), (2) are obvious. To see (3)

$$P\left(\left(\bigcup_{n=1}^{\infty} E_n\right) \Big| F\right) = \frac{P\left(\left(\bigcup_{n=1}^{\infty} E_n\right) \cap F\right)}{P(F)}$$

$$= \frac{P\left(\bigcup_{n=1}^{\infty} (E_n F)\right)}{P(F)}$$

$$= \sum_{n=1}^{\infty} \frac{P(E_n F)}{P(F)} \quad (\text{since } E_n F \text{ are disjoint})$$

$$= \sum_{n=1}^{\infty} P(E_n|F).$$

Example 1.

A bin contains 3 types of disposable flashlights. The probability that a type 1 flashlight will give more than 100 hours of use is .7, with the corresponding probabilities for type 2 and type 3 flashlights being .4 and .3, respectively. Suppose that 20 percent of the flashlights in the bin are type 1, 30 percent are type 2, and 50 percent are type 3.

(a) What is the probability that a randomly chosen flashlight will give more than 100 hours of use?
(b) Given that a flashlight lasted more than 100 hours, what is the conditional probability that it was a type j flashlight, j = 1, 2, 3?

Solution: Let E be the event that a random chosen flashlight will give more than 100 hours.

Let F_i $(i=1,2,3)$ be the event that a random chosen flashlight is of type i.

We need to find out (a) $P(E)$; and
(b) $P(F_i | E)$.

From the conditions of the question, we know

$$P(E|F_1) = 0.7, \quad P(E|F_2) = 0.4$$

$$P(E|F_3) = 0.3.$$

$$P(F_1) = 0.2, \quad P(F_2) = 0.3, \quad P(F_3) = 0.5.$$

Hence $$P(E) = \sum_{i=1}^{3} P(F_i)\, P(E|F_i)$$

$$= 0.2 \times 0.7 + 0.3 \times 0.4 + 0.5 \times 0.3$$

$$P(F_1|E) = \frac{P(F_1) \cdot P(E|F_1)}{P(E)}$$

$$= \frac{0.2 \times 0.7}{0.2 \times 0.7 + 0.3 \times 0.4 + 0.5 \times 0.3}.$$

$$= \frac{14}{41}$$

Similarly

$$P(F_2|E) = \frac{12}{41}, \quad P(F_3|E) = \frac{15}{41}. \quad \square$$

§ 3.3. Independent events.

Let E, F be two events. In general, knowing that F has occurred changes the chance of E's occurrence, that is, possibly $P(E|F) \neq P(E)$.

If $P(E|F) = P(E)$, we say E is independent of F.

Notice that

$$P(E|F) = P(E) \iff \frac{P(EF)}{P(F)} = P(E)$$

$$\iff P(EF) = P(E) \cdot P(F)$$

$$\iff P(F|E) = P(F)$$

Def. We say that E and F are independent if

$$P(EF) = P(E) \cdot P(F).$$

Example 2. A card is randomly chosen from a deck of 52 playing cards.

E — the event that the chosen card is an Ace "A"

F — the event that the chosen card is a spade. "♠"

Determine whether or not E and F are independent.

Solution:

$$P(E) = \frac{4}{52}, \quad P(F) = \frac{13}{52} = \frac{1}{4}.$$

$$P(EF) = \frac{1}{52} = P(E)\,P(F).$$

Hence E, F are independent.

Prop 3. If E and F are independent, then

(1) E and F^c are independent

(2) E^c and F^c are independent.

Pf. (1)

$$P(E \cap F^c) = P(E) - P(EF)$$

$$= P(E) - P(E) \cdot P(F)$$

$$= P(E)\big(1 - P(F)\big)$$

$$= P(E)\,P(F^c),$$

Hence E, F^c are independent.

(2) can be obtained from (1).

- Independence of 3 or more events.

Def. We say 3 events E, F G are independent if

(1) $P(EFG) = P(E)P(F)P(G)$.

(2) $P(EF) = P(E) \cdot P(F)$

$P(EG) = P(E)P(G)$

$P(FG) = P(F)P(G)$.

Def. Let $E_1, E_2, \cdots, E_n$ be a finite family of events.

Say $E_1, \cdots, E_n$ are independent if for any sub-collection

$E_{i_1}, E_{i_2}, \cdots, E_{i_r}$ (with $i_1, \cdots, i_r$ being distinct),

$$P(E_{i_1} E_{i_2} \cdots E_{i_r}) = P(E_{i_1}) \cdots P(E_{i_r}).$$

Def: We say an infinite family of events are independent if every finite subfamily of them is independent.

Def. (Independence of sub-experiments).

An experiment might consist of some sub-experiments,

For instance, the experiment that rolling a coining continuously consists of a sequence of sub-experiments, where the n-th subexperiment is the n-th toll of the coin, $n=1,2,\cdots$.

We say these sub-experiments are independent if

$E_1\ E_2, \cdots, E_n$ are independent

whenever E_i is an event whose occurence depends only on the i-th sub-experiment.

These sub-experiments are said to be trials if the set of possible outcomes of each sub-experiment are the same.

Chap 4. Random Variables.

§4.1 Introduction to random Variables.

Def. For a random experiment, a random Variable (r.v.) X is a real-valued function defined on the sample space S. That is,

$X : S \to \mathbb{R}$ is a function.

Example 1. Flip 3 fair coins. Let X be the number of the heads that appear.

$$X = \#\{\text{heads that appear}\}$$

Eg. if the outcome is (T, H, T), then $X = 1$

if the outcome is (H, T, H), then $X = 2$.

Example 2. Two fair dice are rolled.

Y = the product of the numbers that appear.

If the outcome is $(2, 5)$, then $Y = 10$.

Example 3.

Z = the life-time (in hours) of a randomly chosen flashlight.

§ 4.2. Discrete random variables.

Def. A r.v. X is said to be discrete if X can take on at most countably many different values.

Def. For a discrete r.v. X, the prob. mass function of X is defined by

$$p(a) = P\{X = a\}$$
$$= P\{\omega \in S : X(\omega) = a\},$$
$$\forall\, a \in \mathbb{R}.$$

Example 4: $X = \#\{$heads that appear in rolling 3 fair coins$\}$

$$\{X=0\} = \{(T,T,T)\}$$
$$\{X=1\} = \{(H,T,T), (T,H,T), (T,T,H)\}$$
$$\{X=2\} = \{(H,H,T), (H,T,H), (T,H,H)\}.$$
$$\{X=3\} = \{(H,H,H)\}.$$

So $\quad p(0) = \frac{1}{8}, \quad p(1) = \frac{3}{8}, \quad p(2) = \frac{3}{8}, \quad p(3) = \frac{1}{8}$

and $p(a) = 0$ for all $a \notin \{0, 1, 2, 3\}$.