Math 3280 22-09-15

Review.

- Axiomatic approach to probability.

A prob. P on the sample space S satisfies:

Axiom I: $0 \leq P(E) \leq 1$, $\forall$ any Event E

Axiom II: $P(S) = 1$.

Axiom III: If $E_1, E_2, \cdots$, is a sequence of events which are mutually exclusive, then $P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$

(Countable additivity of prob.).

- Properties derived from the above axioms:

  - Inclusive-exclusive identity.

$$P(E \cup F) = P(E) + P(F) - P(EF).$$

$$P\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1 < i_2 < \cdots < i_r} P(E_{i_1} E_{i_2} \cdots E_{i_r}).$$

• (Countable sub-additivity of Prob)

$$P\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} P(E_k).$$

Prop. (Continuity of Prob.)

① $P\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} P(E_n)$ if $E_1 \subset E_2 \subset \cdots$

② $P\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} P(E_n)$ if $E_1 \supset E_2 \supset \cdots$

Pf. We first prove ①.

Write $F_1 = E_1$

$F_2 = E_2 \setminus E_1$

$\cdots$

$F_n = E_n \setminus \bigcup_{i=1}^{n-1} E_i$

$\cdots$

Then $F_1, \cdots, F_n, \cdots$ are mutually exclusive.

and $\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i = E_n$

$\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$

Hence

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = P\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} P(F_n)$$ (since (F_n) are mutually disjoint)

$$= \lim_{n\to\infty} \sum_{k=1}^{n} P(F_k)$$

$$= \lim_{n\to\infty} P(F_1 \cup \cdots \cup F_n)$$ (since $F_1, \cdots, F_n$ are disjoint)

$$= \lim_{n\to\infty} P(E_n).$$

Next we prove ②.

Notice that

$$E_1^c \subset E_2^c \subset \cdots$$

By ①,

$$P\left(\bigcup_{n=1}^{\infty} E_n^c\right) = \lim_{n\to\infty} P(E_n^c).$$

But

$$\text{LHS} = 1 - P\left(\bigcap_{n=1}^{\infty} E_n\right),$$

$$\text{RHS} = \lim_{n\to\infty} 1 - P(E_n).$$

This implies that $$P\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} P(E_n).$$

Example 1. If $P(E) = 0.8$, $P(F) = 0.9$

Show that $P(E \cap F) \geq 0.7$.

Pf. Recall

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

Hence

$$\begin{aligned} P(E \cap F) &= P(E) + P(F) - P(E \cup F) \\ &= 0.8 + 0.9 - P(E \cup F) \\ &\geq 0.8 + 0.9 - 1 = 0.7. \quad \blacksquare \end{aligned}$$

Example 2.

If $P(E) = 0.8$, $P(F) = 0.9$, $P(E \cap F) = 0.75$

find the probability that exactly one of E or F occurs.

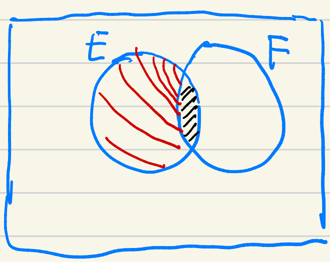
Solution: Let G denote the event that exactly one of E or F occurs.

$$G = (E \setminus F) \cup (F \setminus E)$$

(disjoint union)

Hence $P(G) = P(E \setminus F) + P(F \setminus E)$.

Notice that $E = (E \setminus F) \cap (E \cap F)$



Hence $P(E) = P(E \setminus F) + P(E \cap F) \Rightarrow P(E \setminus F) = P(E) - P(E \cap F)$

$$= 0.8 - 0.75$$

$$= 0.05.$$

Similarly,

$$P(F \setminus E) = P(F) - P(E \cap F)$$
$$= 0.9 - 0.75$$
$$= 0.15$$

Hence

$$P(G) = P(E \setminus F) + P(F \setminus E)$$
$$= 0.05 + 0.15$$
$$= 0.20.$$

$\boxed{//}$

§2.6 Sample space having equally likely outcomes.

In many experiments, it is natural to assume that all outcomes have the same chance to occur.

In this case,

$$P(E) = \frac{\#\text{ of outcomes in } E}{\#\text{ of outcomes in } S} = \frac{\# E}{\# S}.$$

<u>Example 3.</u> If two dices are rolled, what is the prob. that the sum of two outcomes is equal to 6 ?

Solution: Let E be the event that the sum of two outcomes is equal to 6. Then

$$E = \{(i,j) : i, j \in \{1, 2, \cdots, 6\}, i+j=6\}$$
$$= \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$$

and

$$S = \{(i,j) : i, j \in \{1, 2, \cdots, 6\}\}$$

Hence $P(E) = \frac{\#E}{\#S} = \frac{5}{36}$. $\blacksquare$

Example 4.

A committee of 5 is to be selected from a group of 6 men and 9 women. If the
selection is made randomly, what is the probability that the committee consists of 3
men and 2 women?

Solution: Let E denote the event that the selected committee consists of 3 men and 2 women.

Let S be whole sample space.

Then

$$\# S = \binom{15}{5},$$

$$\# E = \binom{6}{3} \cdot \binom{9}{2}.$$

Hence $$P(E) = \frac{\# E}{\# S} = \frac{\binom{6}{3}\binom{9}{2}}{\binom{15}{5}}.$$

$$\binom{n}{m} = \frac{n!}{m!\,(n-m)!}. \quad \left(\begin{array}{l} n! = n \times (n-1) \times \cdots 1 \\ 0! = 1 \end{array} \right)$$

Example 5.

In the game of bridge, the entire deck of 52 cards is dealt out to 4 players. What is the probability that
(a) one of the players receives all 13 spades;
(b) each player receives 1 ace?

(a)
Solution: Let E be the event that one of the players receives all 13 spades.

Let E_i be the event that the i-th player receives all 13 spades, $i=1,2,3,4$.

$E = \bigcup_{i=1}^{4} E_i$, $E_1, \cdots, E_4$ are mutually exclusive.

So $P(E) = P(E_1) + P(E_2) + P(E_3) + P(E_4)$.

$$\# E_1 = \binom{39}{13} \cdot \binom{26}{13} \cdot \binom{13}{13}$$

Similarly, $\# E_2 = \# E_3 = \# E_4 = \# E_1$.

$$\# S = \binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}$$

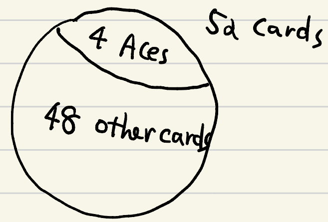
Hence
$$P(E_1) = \frac{\# E_1}{\# S} = \frac{\binom{39}{13}\binom{26}{13}}{\binom{52}{13}\binom{39}{13}\binom{26}{13}} = \frac{1}{\binom{52}{13}}$$

So is $P(E_i)$, $i = 2, 3, 4$.

$$P(E) = \sum_{i=1}^{4} P(E_i) = \frac{4}{\binom{52}{13}}.$$

(b)

Let F be the event that each player receives an Ace.



$$\#F = \binom{4}{1}\cdot\binom{48}{12}\cdot\binom{3}{1}\cdot\binom{36}{12}\cdot\binom{2}{1}\binom{24}{12}$$

Hence $$P(F) = \frac{\binom{4}{1}\binom{48}{12}\binom{3}{1}\binom{36}{12}\binom{2}{1}\binom{24}{12}}{\binom{52}{13}\binom{39}{13}\binom{26}{13}}.$$