Introductory Probability 22-09-08

Chapter 2 Axioms of probability

1. Introduction.

- Probability is a math area dealing with random behaviors.
- It has a history of more than 300 years in the study.
- It came from gambling in the early stage, and gamings of chance.

2. Random experiments, outcomes, sample space, events.

Random experiments / outcomes.

Example: ① Toss a coin to get a head or a tail.

② Roll a dice to see the number of the top face.

③ Measure the height of a randomly chosen student in the campus.

Def. (Sample space). The set of all possible outcomes of an experiment is called the sample space of the experiment.

Usually, we use S to denote the sample space.

Example ① Toss a coin once.

$$S = \{H, T\}.$$

Toss a coin twice.

$$S = \{HH, HT, TH, TT\}$$

② Roll a dice once

$$S = \{1, 2, 3, 4, 5, 6\}.$$

Roll a dice 3 times.

$$S = \{(i, j, k) : \quad i, j, k \in \{1, 2, 3, 4, 5, 6\}\}.$$

③ height of a randomly chosen student (in meters)

$$S = \{0 < x < \infty\} = (0, \infty)$$

Def (event) Let S be the sample space of an experiment.

Every subset E of S is called an event.

If an outcome of the experiment is contained in the event E, then we say that E has occured.

- Basic operations on events.

  Union: $E \cup F$

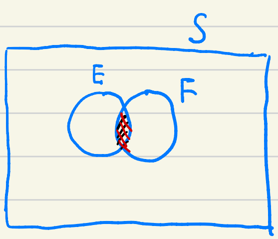
  Intersection: $E \cap F$

  Complement $E^c = S \setminus E$

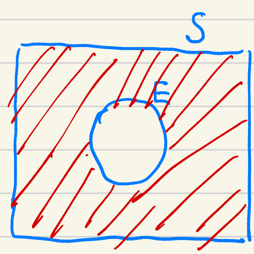
- $\emptyset$ Null event.

  We say two events E, F are mutually exclusive if $E \cap F = \emptyset$.

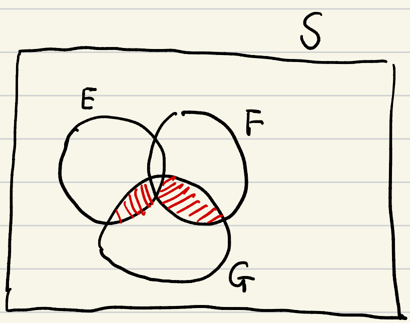
- Venn diagram:



$E \cap F$



E^c



$$\left((E \cap G) \cup (F \cap G)\right) \setminus (E \cap F \cap G)$$

- Laws.

(i) $E \cup F = F \cup E, \quad E \cap F = F \cap E$ — Commutative laws

$E \cap (F \cup G) = (E \cap F) \cup (E \cap G)$ — distributive law

$$\left.\begin{aligned} E \cup (F \cup G) &= (E \cup F) \cup G \\ E \cap (F \cap G) &= (E \cap F) \cap G. \end{aligned}\right\}$$ associative laws

(ii) De Morgan's laws

$$\left(\bigcup_{n=1}^{\infty} E_n \right)^c = \bigcap_{n=1}^{\infty} E_n^c$$

$$\left(\bigcap_{n=1}^{\infty} E_n \right)^c = \bigcup_{n=1}^{\infty} E_n^c.$$

Pf. Let us prove the first equality in (ii)

$$x \in \left(\bigcup_{n=1}^{\infty} E_n \right)^c$$

$$\Longleftrightarrow \quad x \in S, \quad x \notin \bigcup_{n=1}^{\infty} E_n$$

$$\Longleftrightarrow \quad x \in S, \quad x \notin E_n \text{ for } n = 1, 2, \cdots$$

$$\Longleftrightarrow \quad x \in E_n^c \text{ for } n = 1, 2, \cdots$$

$$\Longleftrightarrow \quad x \in \bigcap_{n=1}^{\infty} E_n^c$$

Hence $\left(\bigcup_{n=1}^{\infty} E_n \right)^c = \bigcap_{n=1}^{\infty} E_n^c.$ ▨

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§2.3. Axioms of probability.

Q: How can we define the prob. of an event?

An intuitive approach:

repeat the random experiment n times.

Let $n(E)$ be the times that an event E occurs

Let $$P(E) = \lim_{n\to\infty} \frac{n(E)}{n}.$$

Drawbacks : ① why does the limit exist?

② Even if the limit exist, why is it independent of the experiments?

The axiomatic approach to prob. (by Kolmogrov)

Def. (Prob. of an event).

Let S be the sample space of a random experiment.

A probability P on S is a function that assigns a value to each event E such that the following 3 axioms hold:

Axiom 1 : $0 \leq P(E) \leq 1$, $\forall$ event E.

Axiom 2 : $P(S) = 1$.

Axiom 3 : If $E_1, E_2, \dots$, are a sequence of events which are mutually exclusive,

then $$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$$

(Countable additivity of prob.)

§ 2.4. Some properties of probability.

Prop 1. $P(\emptyset) = 0$.

Pf. Let $E_1 = S$, and $E_n = \emptyset$ for $n = 2, 3, \cdots$.

Then $E_1, E_2, \cdots,$ are mutually exclusive.

By Axiom 3,

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$$
$$= P(E_1) + P(E_2) + \cdots$$
$$= P(S) + P(\emptyset) + P(\emptyset) + \cdots$$

LHS ≤ 1, RHS ≤ 1 only occurs when $P(\emptyset) = 0$. $\boxtimes$

Prop 2. $P(E^c) = 1 - P(E)$.

Pf. Notice that

$$S = E^c \cup E \cup \phi \cup \phi \cdots$$

By Axiom 3 and Prop 1,
Axiom 2

$$1 = P(S) = P(E^c) + P(E).$$

$\boxed{/\!/}$

Prop 3. Let E, F be two events. Then

$$P(E \cup F) = P(E) + P(F) - P(E \cap F).$$

Pf.

$$E \cup F = E \cup (F \setminus E)$$

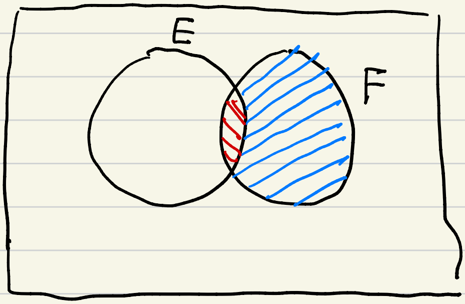
since $E \cap (F \setminus E) = \phi$, so by Axiom 3,

$$P(E \cup F) = P(E) + P(F \setminus E). \quad ①$$

Now we consider $P(F \setminus E)$.

Notice that

$$F = (F \setminus E) \cup (E \cap F)$$



red $\leftrightarrow$ $E \cap F$

blue $\leftrightarrow$ $F \setminus E$.

Using Axiom 3 again,

$$P(F) = P(F \setminus E) + P(E \cap F)$$

hence

$$P(F \setminus E) = P(F) - P(E \cap F) \quad ②$$

Plugging ② into ① yields the desired identity. $\boxed{///}$

Prop 4. (Inclusion-exclusion identity).

$$P(E_1 \cup \cdots \cup E_n) = \sum_{i=1}^{n} P(E_i) - \sum_{i_1<i_2} P(E_{i_1} \cap E_{i_2})$$

$$+ \sum_{i_1<i_2<i_3} P(E_{i_1} E_{i_2} E_{i_3}) - \cdots + (-1)^{n+1} P(E_1 E_2 \cdots E_n)$$

$$= \sum_{r=1}^{n} (-1)^{r+1} \cdot \sum_{i_1<\cdots<i_r} P(E_{i_1} E_{i_2} \cdots E_{i_r})$$

Pf. By induction on n.

By prop 3, the identitity holds for $n=2$.

Next suppose the identity holds for $n=k$.

Then

$$P(E_1 \cup \cdots \cup E_k \cup E_{k+1})$$

$$= P((E_1 \cup \cdots \cup E_k) \cup E_{k+1})$$

$$= P(E_1 \cup \cdots \cup E_k) + P(E_{k+1})$$

$$- P\left((E_1 E_{k+1}) \cup (E_2 E_{k+1}) \cdots \cup (E_k E_{k+1}) \right)$$

Using induction on $n=k$

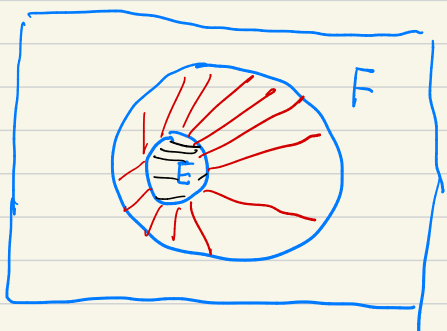
$= \ldots\ldots$

$=$ desired sum. $\blacksquare$

Prop 5. Suppose $E \subset F$. Then

$$P(E) \leq P(F).$$

Pf. $F = E \cup (F \setminus E)$.



By Axiom 3, $P(F) = P(E) + P(F \setminus E)$

Notice by Axiom 1, $P(F \setminus E) \geq 0$,

so $P(F) \geq P(E)$.

$\blacksquare$.

Prop 6. Let $E_1, E_2, \cdots,$ be a sequence of events.

Then

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} P(E_n).$$

(Countable sub-additivity of prob.)

Proof. First we write $\bigcup_{n=1}^{\infty} E_n$ as the union of some disjoint events. To do so, write

$$F_1 = E_1$$

$$F_2 = E_2 \setminus E_1$$

$$F_3 = E_3 \setminus (E_1 \cup E_2),$$

. . . .

$$F_n = E_n \setminus \left(\bigcup_{i=1}^{n-1} E_i \right),$$

. . . .

Then

- $F_n \subset E_n, \quad n = 1, \cdots,$
- $\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i \qquad (*)$
- $\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i$
- $F_1, F_2, \cdots$ are mutually exclusive. ✓

To show $(*)$, recall that $F_i \subset E_i$ so

$$\bigcup_{i=1}^{n} F_i \subset \bigcup_{i=1}^{n} E_i.$$

To prove $\bigcup_{i=1}^{n} F_i \supset \bigcup_{i=1}^{n} E_i$,

let $x \in \bigcup_{i=1}^{n} E_i$. Then $x \in E_i$ for some $i \leq n$.

Let i be the smallest integer $\leq n$ such that

$$x \in E_i$$

Then
$$x \in E_i \setminus \bigcup_{j=1}^{i-1} E_j = F_i$$

which means

$$\bigcup_{i=1}^{n} E_i \subset \bigcup_{i=1}^{n} F_i,$$

which proves $(*)$

Now using Axiom 3 to $P\left(\bigcup_{n=1}^{\infty} F_n\right)$

we have

$$P\left(\bigcup_{n=1}^{\infty} F_n\right) = \sum_{n=1}^{\infty} P(F_n)$$

$$\leq \sum_{n=1}^{\infty} P(E_n),$$

and we are done since

$$P\left(\bigcup_{n=1}^{\infty} F_n\right) = P\left(\bigcup_{n=1}^{\infty} E_n\right). \quad \blacksquare$$