Solution to Exercise 3

1. (a) $\text{int}\left(I^2\right) = \emptyset$, $\text{ri}\left(I^2\right) = \{(x,y,0) \mid 0 < x,y < 1\}$, $\text{int}\left(I^3\right) = \text{ri}\left(I^3\right) = \{(x,y,z) \mid 0 < x,y,z < 1\}$
(b) follows directly from (a).

2.
Suppose the condition holds for x. Let $\bar{x} \in \text{ri}(C)$. If $x = \bar{x}$, then we are done. So assume $x \neq \bar{x}$. Then there exists $\gamma > 0$ such that $y = x + \gamma(x - \bar{x}) \in C$. Hence $x = \frac{1}{1+\gamma}y + \frac{\gamma}{1+\gamma}\bar{x}$. Since $\bar{x} \in \text{ri}(C)$, $y \in C$, by the line segment principle, we have $x \in \text{ri}(C)$. The other direction is clear from the fact that $x \in \text{ri}(C)$.

3. For any $i = 1, \ldots, k$, any $x, y \in \text{dom} f_i$, and $\lambda \in [0,1]$, we have
$f_i(\lambda x + (1-\lambda)y) \leq \lambda f_i(x) + (1-\lambda) f_i(y)$
Then it follows that

$$f(\lambda x+(1-\lambda)y) = \sum_{i=1}^{k} w_i f_i(\lambda x+(1-\lambda)y) \leq \sum_{i=1}^{k} w_i \left(\lambda f_i(x) + (1-\lambda) f_i(y)\right) = \lambda f(x)+(1-\lambda)f(y)$$

Hence, $f(x)$ is convex.