Solution to Exercise 5

1. (a) Assume that f is convex. Let (x,u) and (y,v) be two points of the epigraph: $u \geq f(x)$ and $v \geq f(y)$. In particular, $(x,y) \in \mathrm{dom}(f)^2$. Let $t \in]0,1[$. Then $f(tx+(1-t)y) \leq tu+(1-t)v$. Thus, $t(x,u)+(1-t)(y,v) \in \mathrm{epi}(f)$, which proves that $\mathrm{epi}(f)$ is convex.

Conversely, assume that $\mathrm{epi}(f)$ is convex. Let $(x,y) \in \mathrm{dom}(f)^2$. For (x,u) and (y,v) two points in $\mathrm{epi}(f)$, and $t \in [0,1]$, the point $t(x,u)+(1-t)(y,v)$ belongs to $\mathrm{epi}(f)$. So, $f(t(x+(1-t)y) \leq tu+(1-t)v$.

- If $f(x)$ et $f(y)$ are $> -\infty$, we can choose $u = f(x)$ and $v = f(y)$, which gives the convexity of f.

- If $f(x) = -\infty$, we can choose u arbitrary close to $-\infty$. Letting u go to $-\infty$, we obtain $f(t(x+(1-t)y) = -\infty$, which again gives the convexity of f.

(b) We only need to prove $\mathrm{epi}\left(\sup_{i\in I} f_i\right) = \bigcap_{i\in I} \mathrm{epi}\left(f_i\right)$. Then the result follows directly from (a) and the fact that the intersection of any number of convex sets is convex.

In fact, for any $(x,t) \in \mathrm{epi}\left(\sup_{i\in I} f_i\right)$, $\sup_{i\in I} f_i(x) \leq t$. Hence, $f_i(x) \leq \sup_{i\in I} f_i(x) \leq t, \forall i \in I$. Then $(x,t) \in \mathrm{epi}\left(f_i\right), \forall i \in I$, i.e., $(x,t) \in \bigcap_{i\in I} \mathrm{epi}\left(f_i\right)$. On the other hand, for any $(x,t) \in \bigcap_{i\in I} \mathrm{epi}\left(f_i\right)$, $t \geq f_i(x), \forall i \in I$. This implies $t \geq \sup_{i\in I} f_i(x)$ and (hence) $(x,t) \in \mathrm{epi}\left(\sup_{i\in I} f_i\right)$.

2. (a) For all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$ and $\lambda \in [0,1]$, we have

$$\begin{aligned}
g(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) &= f(\lambda(\boldsymbol{A}\boldsymbol{x}+\boldsymbol{b}) + (1-\lambda)(\boldsymbol{A}\boldsymbol{y}+\boldsymbol{b})) \\
&\leq \lambda f(\boldsymbol{A}\boldsymbol{x}+\boldsymbol{b}) + (1-\lambda) f(\boldsymbol{A}\boldsymbol{y}+\boldsymbol{b}) \\
&= \lambda g(\boldsymbol{x}) + (1-\lambda) g(\boldsymbol{y})
\end{aligned}$$

(b) From the convexity of g, $g(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) \leq \lambda g(\boldsymbol{x}) + (1-\lambda)\boldsymbol{y}$. Since h is non-decreasing, we have

$$h(g(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y})) \leq h(\lambda g(\boldsymbol{x}) + (1-\lambda)\boldsymbol{y}).$$

Hence

$$\begin{aligned}
f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y}) &= h(g(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{y})) \\
&\leq h(\lambda g(\boldsymbol{x}) + (1-\lambda) g(\boldsymbol{y})) \\
&\leq \lambda h(g(\boldsymbol{x})) + (1-\lambda) h(g(\boldsymbol{y})) \\
&= \lambda f(\boldsymbol{x}) + (1-\lambda) f(\boldsymbol{y})
\end{aligned}$$

3. (a) is false. Consider $f(x) = x^4$, which is strictly convex on $\mathbb{R}$ but $f''(0) = 0$.

(b), (c) and (d) are true.