Assignment 6, Due 11:59 pm, Thursday Dec 1, 2022

Please upload your assignment to the Blackboard of this course

(1) Let $\alpha(t)$ be a regular curve on a regular surface M, where t may not be proportional to arc length. Let $\alpha' = \frac{d\alpha}{dt}$, etc. Suppose $(\alpha'')^T = \lambda(t)\alpha'$ for some smooth function $\lambda(t)$ on α. Prove that after reparametrization, α is a geodesic. That is: Find a new parameter $\tau = f(t)$ so that $(\ddot{\alpha})^T = 0$, where $\dot{\alpha} = \frac{d\alpha}{d\tau}$ etc.

(2) Consider the hyperbolic plane: $M = \{(x, y) \in \mathbb{R}^2 | \; y > 0\}$. The first fundamental form is:

$$g_{ij} = \frac{1}{y^2}\delta_{ij}.$$

(i) Use the Euler-Lagrange equation for the energy functional to find the geodesic equations.

(ii) Show that the line x=constant, i.e., $\alpha(t) = (c, t)$ where c is a constant is a pre-geodesic.

(iii) Show that the semi-circle $\alpha(t) = (R\cos t, R\sin t)$, $R > 0$ is a constant and $0 < t < \pi$ is a pre-geodesic.

(3) Write down the differential equations for the geodesics on the torus:

$$\mathbf{X}(u, v) = ((a + r\cos v)\cos u, (a + r\cos v)\sin u, r\sin v)$$

with $a > r > 0$. Also, show that if α is a geodesic start at a point on the topmost parallel $(a\cos u, a\sin u, r)$ and is tangent to this parallel, then α will stay in the region with $-\pi/2 \le v \le \pi/2$.

Find also the geodesic curvature of the topmost parallel.

(4) Let $\mathbf{X} : U \to M$, $(u_1, u_2) \to \mathbf{X}(u_1, u_2)$, be a coordinate parametization, with U being an open set in $\mathbb{R}^2$. Suppose the first fundamental form in this coordinate satisfies $g_{12} = 0$, and $g_{11} = g_{22} = \exp(2f)$ for some smooth function f, i.e. $g_{ij} = \exp(2f)\delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and is zero if $i \ne j$. Let $\mathbf{e_1} = \mathbf{X}_1/|\mathbf{X}_1|$, $\mathbf{e_2} = \mathbf{X}_2/|\mathbf{X}_2|$, and $\mathbf{N} = \mathbf{e_1} \times \mathbf{e_2}$. Let $\alpha(s)$ be a geodesic on M such that $\alpha(s) = \mathbf{X}(u_1(s), u_2(s))$. Let $\theta(s)$ be a smooth function on s such that $\alpha'(s) = \mathbf{e_1}(s)\cos\theta(s) + \mathbf{e_2}(s)\sin\theta(s)$, where $\mathbf{e_i}(s) = \mathbf{e_i}(\alpha(s))$. Show that

$$\mathbf{a} := \mathbf{N} \times \alpha' = -\mathbf{e_1}(s)\sin\theta(s) + \mathbf{e_2}(s)\cos\theta(s).$$

Show also that

$$\begin{aligned} k_g &= -\langle \alpha', \mathbf{a}' \rangle \\ &= \exp(-2f)\langle \frac{d}{ds}\mathbf{X}_1, \mathbf{X}_2 \rangle + \theta' \\ &= \left(-u' \frac{\partial f}{\partial v} + v' \frac{\partial f}{\partial u} \right) + \theta'. \end{aligned}$$

(You may use the fact that $\Gamma_{ij}^k = \delta_{ki} f_j + \delta_{kj} f_i - \delta_{ij} f_k$.)