Differentials of maps

Let M_1, M_2 be two regular surfaces and let $F : M_1 \to M_2$ be a smooth map. The differential dF_p at p is a linear map from $T_p(M_1)$ to $T_p(M_2)$ where $q = F(p)$ which is defined as follows. Let $v \in T_p(M)$ and let $\alpha(t)$ be a curve on M_1 with $\alpha(0) = p$, $\alpha'(0) = v$. Then $dF_p(v) := \frac{d}{dt} F(\alpha(t))\Big|_{t=0}$. dF is well-defined, linear and smooth.

Gauss map

Let M be an orientable regular surface and let $\mathbf{N}$ be a unit normal vector field. We also denote the Gauss map by $\mathbf{N}$. That is $\mathbf{N} : M \to \mathbb{S}^2$ which is the unit sphere in $\mathbb{R}^3$. At $q \in \mathbf{N}(p) \in \mathbb{S}^2$, we use the unit normal vector $\mathbf{N}(p)$ and we identify $T_p(M)$ to $T_q(\mathbb{S}^2)$. Let $\mathbf{X}(u^1, u^2)$ $((u^1, u^2) \in U \subset \mathbb{R}^2)$ be a parametrization of M with orientation determined by $\mathbf{N}$. Then $\mathbf{N} : U \to \mathbb{S}^2$, where $\mathbf{N}(u^1, u^2) = \mathbf{N}(\mathbf{X}(u^1, u^2))$. Then $d\mathbf{N} = -\mathcal{S}$. If The Gaussian curvature is nonzero at a point p, then $\mathbf{N}$ can be considered as a parametrization of $\mathbb{S}^2$ near q.

Area of Gauss image

Proposition

Let $p \in M$. *Suppose* $K(p) \neq 0$. *Let* B_n *be a sequence of open sets with* $B_n \to p$ *in the sense that* $\sup_{q \in B_n} |p - q| \to 0$ *as* $n \to \infty$. *Let* A_n *be the area of* B_n *and* $\widetilde{A}_n$ *be the area of the Gauss image* $\mathbf{N}(B_n)$ *of* B_n. *Then*

$$\lim_{n \to \infty} \frac{\widetilde{A}_n}{A_n} = |K(p)|.$$

Proof

Proof.

May assume that B_n is the image of $U_n \subset U$ of the parametrization $\mathbf{X}$, so that $p \leftrightarrow (0,0)$. Then $U_n \to (0,0)$ if $B_n \to p$. So

$$A_n = \iint_{U_n} |\mathbf{X}_1 \times \mathbf{X}_2| du^1 du^2,$$

$$\widetilde{A}_n = \iint_{U_n} |\mathbf{N}_1 \times \mathbf{N}_2| du^1 du^2.$$

Now $d\mathbf{N} = -\mathcal{S}$, so $\mathbf{N}_1 \times \mathbf{N}_2 = \det(-\mathcal{S})\mathbf{X}_1 \times \mathbf{X}_2 = K\mathbf{X}_1 \times \mathbf{X}_2$. Hence

$$\frac{\widetilde{A}_n}{A_n} = \frac{\iint_{U_n} |K||\mathbf{X}_1 \times \mathbf{X}_2| du^1 du^2}{\iint_{U_n} |\mathbf{X}_1 \times \mathbf{X}_2| du^1 du^2} \to |K(p)|.$$

□

Meaning of $K > 0, K < 0$

Since $K = \det(\mathcal{S}) = \det(-d\mathbf{N})$, $K > 0$ means **N** is orientation preserving, and $K < 0$ means orientation reversing.
Hence

$$\iint_M K dA$$

can be considered as the signed area of the Gauss image.

Examples

- M is a plane. The Gauss image is a point and the Gaussian curvature is zero. The area of the Gauss image is zero.
- Let M be the circular cylinder. The Gaussian curvature is zero. That Gauss image is a circle. The area of the Gauss image is zero.
- Let M be the sphere of radius R. The Gaussian curvature if $1/R^2$. The Gauss image is the whole unit sphere. So the area of the Gauss image is 4π.
- Let M be the torus. Then

$$K = \frac{\cos u}{r(a + r\cos u)}.$$

Hence

$$\iint_M KdA = \int_0^{2\pi}\int_0^{2\pi} \frac{\cos u}{r(a + r\cos u)} \cdot r(a + r\cos u)dudv = 0.$$