Orientation of regular surfaces

Definition

Let M be a regular surface in $\mathbb{R}^3$. M is said to be orientable if there is a unit vector field $\mathbf{N}$ on M such that $\mathbf{N}$ is smooth; $\mathbf{N}$ has unit length; $\mathbf{N}$ is orthogonal to $T_p(M)$ at all point. If such $\mathbf{N}$ exists, then it is called an orientation of M.

Basic facts

- If $\mathbf{N}$ is an orientation, then $-\mathbf{N}$ is also an orientation. There are exactly two orientations on an orientable surface.
- $\mathbf{N}$ is smooth means that if $\mathbf{N} = (N_1, N_2, N_3)$ then each N_i is a smooth function.
- $\mathbf{N}$ is continuous and satisfies (ii), (iii) above that $\mathbf{N}$ is smooth.

An instrinsic definition

We have the following intrinsic characterization of orientable surface.

Proposition

M is orientable if and only if there exist coordinate charts covering M so that the change of coordinate matrices have positive determinant.

Proof

Proof: (Sketch) If M is orientable and $\mathbf{N}$ is an orientation. Let $(\mathbf{X}_\alpha, U_\alpha)$ be coordinate charts covering M. If the coordinates of U_α are denoted by (u, v), then we may choose (u, v) so that

$$\mathbf{N} = \frac{(\mathbf{X}_\alpha)_u \times (\mathbf{X}_\alpha)_v}{|(\mathbf{X}_\alpha)_u \times (\mathbf{X}_\alpha)_v|}. \text{ (Why?)}$$

Then these are the coordinate charts we want.

Proof, cont.

Conversely, if $(\mathbf{X}_\alpha, U_\alpha)$ be coordinate charts covering M so that the change of coordinate matrices have positive determinant. Define $\mathbf{N}$ as above, then this gives an orientation of M. (Why?)

The shape operator

Let M be a regular surface in $\mathbb{R}^3$. Suppose M is orientable with orientaion $\mathbf{N}$. That is:

- $\mathbf{N}$ is smooth;
- $\mathbf{N}$ has unit length;
- $\mathbf{N}$ is orthogonal to $T_p(M)$ at all point.

Definition

The shape operator $\mathcal{S}_p$ with respect to $\mathbf{N}$ at p is the operator defined as follows: Let $\mathbf{v} \in T_p(M)$ and let $\alpha(t)$, $-\epsilon < 0 < \epsilon$ be a smooth curve on M with $\alpha(0) = p$, $\alpha'(0) = \mathbf{v}$. Then $\mathcal{S}_p(\mathbf{v})$ is defined as

$$\mathcal{S}_p(\mathbf{v}) = -\frac{d}{dt}(N(\alpha(t)))\Big|_{t=0}.$$

Remarks

- Notice that there is a negative sign on the RHS in the above.
- $\mathcal{S}_p$ is also called the *Weingarten map* of M at p.
- If $\mathbf{N}$ is a unit normal vector field, then $\mathbf{N}_1 := -\mathbf{N}$ is also a unit normal vector field. The shape operator with respect to $\mathbf{N}_1$ is the negative of the shape operator with respect to $\mathbf{N}$.

Basic facts

Proposition

With the above notation, the following are true:

(i) $\mathcal{S}_p$ *is well-defined.*

(ii) $\mathcal{S}_p$ *is a linear map from* $T_p(M)$ *to* $T_p(M)$.

(iii) $\mathcal{S}_p$ *is self-adjoint with respect to the first fundamental form.*

(vi) $\mathcal{S}$ *is smooth.*

$\mathcal{S}_p$ is well-defined

Proof: (Sketch) Let $\mathbf{X}(u,v)$ be a local parametrization so that $\mathbf{X}(u_0,v_0)=p$. Then $\mathbf{N}=\mathbf{N}(u,v)$.
Let $\alpha(t)=\mathbf{X}(u(t),v(t))$ so that $(u(0),v(0))=(u_0,v_0)$. Then

$$\frac{d\mathbf{N}(\alpha(t))}{dt}=\mathbf{N}_u u'+\mathbf{N}_v v'.$$

Let $\mathbf{v}=a\mathbf{X}_u+b\mathbf{X}_v$. Now $\mathbf{v}=\alpha'(0)=\mathbf{X}_u u'+\mathbf{X}_v v'$, so $u'=a, v'=b$ at p. Hence

$$\frac{d\mathbf{N}(\alpha(t))}{dt}=a\mathbf{N}_u+b\mathbf{N}_v.$$

So $\mathcal{S}_p$ is well-defined.

$\mathcal{S}_p$ is a linear map from $T_p(M)$ to $T_p(M)$

Note that $\mathbf{N}_u, \mathbf{N}_v$ are in $T_p(M)$ (Why?). So $\mathcal{S}_p : T_p(M) \to T_p(M)$. It is also linear. (Why?)

$\mathcal{S}_p$ is self-adjoint

To prove $\mathcal{S}_p$ is self adjoint. Let $\mathbf{v}, \mathbf{w} \in T_p(M)$. Let $\mathbf{v} = a\mathbf{X}_u + b\mathbf{X}_v$, $\mathbf{w} = c\mathbf{X}_u + d\mathbf{X}_v$. Then

$$-\langle \mathcal{S}_p(\mathbf{v}), \mathbf{w} \rangle = \langle a\mathbf{N}_u + b\mathbf{N}_v, c\mathbf{X}_u + d\mathbf{X}_v \rangle$$
$$= ac\langle \mathbf{N}_u, \mathbf{X}_u \rangle + bd\langle \mathbf{N}_v, \mathbf{X}_v + ad\langle \mathbf{N}_u, \mathbf{X}_v \rangle + bc\langle \mathbf{N}_v, \mathbf{X}_u \rangle$$

$$-\langle \mathcal{S}_p(\mathbf{v}), \mathbf{w} \rangle = ac\langle \mathbf{N}_u, \mathbf{X}_u \rangle + bd\langle \mathbf{N}_v, \mathbf{X}_v + cb\langle \mathbf{N}_u, \mathbf{X}_v \rangle + da\langle \mathbf{N}_v, \mathbf{X}_u \rangle$$

So they are equal. (Why?)

Examples of $\mathcal{S}_p$

Let $M = \{ax + by + cz + d = 0\}$. Then we can choose $\mathbf{N} = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}}$. So $\mathcal{S}_p(\mathbf{v}) = \mathbf{0}$. Let $M = \mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}$. $\mathbf{N} = (x, y, z)$. Suppose $\alpha(t) = (x(t), y(t), z(t))$ is a curve on M with $\alpha'(0) = \mathbf{v}$. Then $\mathbf{v} = (x'(0), y'(0), z'(0)$. So $\mathcal{S}_p(\mathbf{v}) = -\frac{d}{dt} N(x(t), y(t), z(t))|_{t=0} = -\mathbf{v}$. And $\mathcal{S}_p = -\mathrm{Id}$.

More examples

- Let $M = \{x^2 + y^2 = 1\}$ the circular cylinder. Parametrize M by $\mathbf{X}(u, v) = (\cos u, \sin u, v)$. Then $\mathbf{X}_u = (-\sin u, \cos u, 0), \mathbf{X}_v = (0, 0, 1)$. We can take $\mathbf{N} = (\cos u, \sin u, 0)$. Then $\mathcal{S}_p(\mathbf{X}_u) = -\mathbf{N}_u = -(-\sin u, \cos u, 0) = -\mathbf{X}_u$. $\mathcal{S}_p(\mathbf{X}_v) = \mathbf{0}$.
- Let M be the hyperboloid $M = \{z = y^2 - x^2\}$. We can parametrize it by $\mathbf{X}(u, v) = (u, v, v^2 - u^2)$. Then $\mathbf{X}_u = (1, 0, -2u), \mathbf{X}_v = (0, 1, 2v)$ and $\mathbf{N} = \frac{1}{(u^2+v^2+\frac{1}{4})^{\frac{1}{2}}}(u, -v, \frac{1}{2})$. At $p = (0, 0, 0) = \mathbf{X}(0, 0)$, and if $\mathbf{X}(u(t), v(t))$ is a curve through p, then $\frac{d\mathbf{N}}{dt} = (2u', 2v', 0)$. So $\mathcal{S}_p(\mathbf{X}_u) = -(2, 0, 0), \mathcal{S}_p(\mathbf{X}_v) = (0, 2, 0)$.