Regular surfaces

Definition

A subset $M \subset \mathbb{R}^3$ is said to be a regular surface if for any $p \in M$, there is an open neighborhood U of p in M, an open set D in $\mathbb{R}^2$ and a map $\mathbf{X} : D \to M \cap U$ such that the following are true:

(rs1) $\mathbf{X}$ *is smooth.*

(rs2) $d\mathbf{X}$ *is full rank:* $\mathbf{X}_u = \frac{\partial \mathbf{X}}{\partial u}$ *and* $\mathbf{X}_v = \frac{\partial \mathbf{X}}{\partial v}$ *are linearly independent, for any* $(u, v) \in D$.

(rs3) $\mathbf{X}$ *is a homeomorphism from* D *onto* $M \cap U$*. (That is:* $\mathbf{X}$ *is bijective,* $\mathbf{X}$ *and* $\mathbf{X}^{-1}$ *are continuous).*

Regular surfaces, cont.

- Let M be a regular surface, a map $\mathbf{X} : D \to V$ where V is an open set of M, satisfying the above conditions.
- $\mathbf{X}$ is called a *parametrization*, and V is called a *coordinate chart (patch, neighborhood)*.
- If $\mathbf{X}(u, v) = p$, then (u, v) are called local coordinates of p.
- So a regular surface is a set M in $\mathbb{R}^3$ which can be covered by a family of coordinate charts.

Example 1: graphs, $z = f(x, y)$

Graphs: Let $M = \{(x, y, z) |\ z = f(x, y), (x, y) \in D \subset \mathbb{R}^2\}$. Then M can be covered by a coordinate chart. We can take $U = D \times \mathbb{R}$. $\mathbf{X}(u, v) = (u, v, f(u, v))$ with $(u, v) \in D$. Check:

(rs1) $\mathbf{X}$ is smooth.

(rs2) $d\mathbf{X}$ is full rank: $\mathbf{X}_u = (1, 0, f_u)$ and $\mathbf{X}_v = (0, 1, f_v)$ are linearly independent, for any $(u, v) \in U$.

(rs3) $\mathbf{X}$ is a homeomorphism from D onto $M \cap U = M$. (That is: $\mathbf{X}$ is bijective, $\mathbf{X}$ and $\mathbf{X}^{-1}$ are continuous) (Why?).

Graphs are regular surfaces

So we have:

Proposition

Let $f : D \to \mathbb{R}$ be a smooth function on an open set $D \subset \mathbb{R}^2$. Then the graph of f defined by the following is a regular surface:

$$\mathrm{graph}(f) = \{(x, y, f(x, y)) | \ (x, y) \in D\}.$$

Example 2: Unit sphere, $\{x^2 + y^2 + z^2 = 1\}$

Unit sphere: $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 |\ x^2 + y^2 + z^2 = 1\}$. Take a point p in the northern (open) hemisphere. Let $U = \{z > 0\}$. $D = \{u^2 + v^2 < 1\}$. Let

$$\mathbf{X}(u, v) = (u, v, \sqrt{1 - u^2 - v^2}).$$

How many similar coordinate charts will cover $\mathbb{S}^2$?

Spherical coordinates

Consider another parametrization.

$$\mathbf{X}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

with $\{(\theta, \varphi) |\ 0 < \theta < \pi, 0 < \varphi < 2\pi\}$. Then

$\mathbf{X}_\theta = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$; $\mathbf{X}_\varphi = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)$.

How many similar coordinate charts will cover $\mathbb{S}^2$?

Stereographic projection

There is still another important parametrization, the Stereographic projection. The unit sphere M is considered as the set $x^2 + y^2 + (z-1)^2 = 1$.

$$\pi : M \setminus \{(0,0,2) = N\} \to \mathbb{R}^2\}$$

so that $N, p, \pi(p)$ are on a straight line. Then $\mathbf{X} : \mathbb{R}^2 \to M \setminus \{N\}$ is a coordinate chart.

$$\mathbf{X}(u,v) = \left(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{2(u^2+v^2)}{u^2+v^2+4} \right).$$

Regular surfaces are graphs locally

Proposition

Let M be regular surface and let $\mathbf{X} : U \to M$ be a coordinate parametrization. Then for any $p = (u_0, v_0) \in U$ there is a open set $V \subset U$ with $p \in V$ such that $\mathbf{X}(V)$ is a graph over an open set in one of the coordinate plane.

Review on inverse function theorem

Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map from an open set U to $\mathbb{R}^n$, $F(\mathbf{x}) = \mathbf{y}(\mathbf{x}) =$ where $\mathbf{x} = (x^1, \ldots, x^n)$, $\mathbf{y} = (y^1, \ldots, y^m)$. Let $\mathbf{x_0} = (x_0^1, \ldots, x_0^n) \in U$. The Jacobian matrix of F at $\mathbf{x}_0$ is the $m \times n$ matrix

$$dF_{\mathbf{x}_0} = \left(\frac{\partial y^i}{\partial x^j}(\mathbf{x}_0)\right).$$

Theorem

(Inverse Function Theorem) *Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map. Suppose $F(\mathbf{x}_0) = \mathbf{y}_0$ and $dF_{\mathbf{x}_0}$ is nonsingular. Then there exist open sets $U \supset V \ni \mathbf{x}_0$ and $W \ni \mathbf{y}_0$, such that F is a diffeomorphism from V to W. That is to say, $F : V \to W$ is bijective and F^{-1} is also smooth on W.*

Proof of the inverse function theorem

Proof:

- May assume that $\mathbf{x}_0 = \mathbf{0} = \mathbf{y}_0$. Let $A = dF_{\mathbf{x}_0}$.
- Then
$$F(\mathbf{x}) = A\mathbf{x} + G(\mathbf{x}),$$
$$G(\mathbf{x}_1) - G(\mathbf{x}_2) = o(|\mathbf{x}_1 - \mathbf{x}_2|) \text{ as } \mathbf{x}_1, \mathbf{x}_2 \to \mathbf{O}.$$
- Hence for any $\epsilon > 0$, we can find $\delta > 0$ such that if $\mathbf{x}_1, \mathbf{x}_2 \in B(\mathbf{0}, \delta) = \{|\mathbf{x}| < \delta\}$, we have ,
$$|F(\mathbf{x}_1) - F(\mathbf{x}_2)| \geq |A(\mathbf{x}_1 - \mathbf{x}_2)| - \epsilon|\mathbf{x}_1 - \mathbf{x}_2|$$
- From this we conclude that F is one-one in $B(\mathbf{0}, \delta)$ if $\epsilon > 0$ is small enough.

Proof (cont.)

- Let $\mathbf{y}_1 \in \mathbb{R}^n$. Want to find $\mathbf{x}$ so that $F(\mathbf{x}) = A\mathbf{x} + G(\mathbf{x}) = y_1$.
- $\exists \mathbf{x}_1, A\mathbf{x}_1 = \mathbf{y}_1$.(?) Inductively, $\exists \mathbf{x}_{n+1}$ with $A\mathbf{x}_{n+1} = \mathbf{y}_1 - G(\mathbf{x}_n)$.
- There is $\rho > 0$ such that if $|\mathbf{y}_1| < \rho$, then $\mathbf{x}_n \in B(\mathbf{O}, \frac{1}{4}\delta)$ and $\mathbf{x}_n \to \mathbf{x} \in \overline{B(\mathbf{O}, \frac{1}{2}\delta)} \subset B(\mathbf{O}, \delta)$. (Why?)

Idea of proof: Regular surfaces are graphs

Proof:

(Sketch) Let $\mathbf{X}(u,v) = (x(u,v), y(u,v), z(u,v))$. May assume that at (u_0, v_0)

$$\det \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \neq 0.$$

Let $(x_0, y_0) = (x(u_0, v_0), y(u_0, v_0))$. By the inverse function theorem, there is a nbh of U_1 of (u_0, v_0) and W of (x_0, y_0) so that $(u,v) \to (x,y)$ has a smooth inverse . Then the image of U_1 under $\mathbf{X}$ is of the form

$$\begin{aligned}(x,y) &\to (u(x,y), v(x,y)) \\ &\to (x(u(x,y)), y(u(x,y)), z(u(x,y), v(x,y))) \\ &= (x, y, f(x,y)).\end{aligned}$$

Regular value of a function

Proposition

Let U be an open set in $\mathbb{R}^3$ and let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Suppose a is a regular value of f. (That is: if $f(x, y, z) = a$, then $\nabla f(x) \neq \mathbf{0}$.) Then

$$M = \{(x, y, z) \in U |\ f(x) = a\}$$

is a regular surface.

Examples: Quadratic surfaces

The unit sphere is given by $\{f = 1\}$ with $f(x, y, z) = x^2 + y^2 + z^2$.
$\nabla f = (2x, 2y, 2z)$ which is not zero if $f = 1$. So 1 is a regular value.
In general: Quadric surfaces.

Proof of level sets of regular values are regular surfaces

Proof:
(Sketch) Let $(x_0, y_0, z_0) \in M$. May assume that $f_z \neq 0$ at this point. Consider the map: $F : U \to \mathbb{R}^3$ defined by $F(x, y, z) = (x, y, f(x, y, z))$. Then the Jacobian matrix is invertible at $p = (x_0, y_0, z_0)$. Let $F(x_0, y_0, z_0) = (u_0, v_0, t_0) = q$, with $t_0 = a$. Then there exist nbh V of p and W of q so that F has a smooth inverse F^{-1}. Now $F^{-1}(u, v, t) = (u, v, g(u, v, t))$. Let $W_1 = \{(u, v)|(u, v, a) \in W\}$. Then for $(x, y, z) \in V \cap M$, $F(x, y, z) = (x, y, a) = (u, v, g(u, v, a))$ and so this set is the graph of over (u, v).

Surfaces of revolution

Let $\alpha(t)$ be a regular curve in the *yz*-plane given by

$$\alpha(u) = (0, y(u), z(u))$$

so that $x(t) > 0$. Consider the surface given by

$$\mathbf{X}(u, v) = (y(u)\cos v, y(u)\sin v, z(u)).$$

Then

$$\mathbf{X}_u = (y' \cos v, y' \sin v, z'); \mathbf{X}_v = (-y \sin v, y \cos v, 0).$$

Torus

Rotating a circle $(y-a)^2+z^2=r^2$ about the z-axis, $a>r>0$.

$$\mathbf{X}(u,v)=((r\cos u+a)\cos v,(r\cos u+a),r\sin u).$$

$0<u<2\pi, 0<v<2\pi)$

$$z^2+\left(\sqrt{x^2+y^2}-a\right)^2=r^2.$$

Let $f=z^2+\left(\sqrt{x^2+y^2}-a\right)^2$, then

$$\nabla f=2(\frac{x\left(\sqrt{x^2+y^2}-a\right)}{\sqrt{x^2+y^2}},\frac{y\left(\sqrt{x^2+y^2}-a\right)}{\sqrt{x^2+y^2}},z)$$

Then ∇f is smooth (why?) and r^2 is a regular value of f (why?).