Geodesic equations of surfaces of revolution

Consider the surface of revolution given by

$$\mathbf{X}(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$$

with $f > 0$. In the following f' means $\frac{df}{dv}$, etc. If there is come confusion, we will write f_v instead, etc.
Consider $u^1 \leftrightarrow u, u^2 \leftrightarrow v$.

$$\begin{cases} g_{11} = & E = \langle \mathbf{X}_u, \mathbf{X}_u \rangle = f^2, ; \\ g_{12} = & g_{21} = F = \langle \mathbf{X}_u, \mathbf{X}_v \rangle = 0 \\ g_{22} = & G = \langle \mathbf{X}_v, \mathbf{X}_v \rangle = (f')^2 + (g')^2. \end{cases}$$

So

$$\begin{cases} \Gamma_{11}^1 = 0, \Gamma_{12}^1 = \frac{f'}{f}, \Gamma_{22}^1 = 0; \\ \Gamma_{11}^2 = -\frac{ff'}{(f')^2+(g')^2}, \Gamma_{12}^2 = 0, \Gamma_{22}^2 = \frac{f'f''+g'g''}{(f')^2+(g')^2}. \end{cases}$$

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Hence geodesic equations are:

$$\begin{cases} \ddot{u} + \frac{2f'}{f}\dot{u}\dot{v} = 0; \\ \ddot{v} - \frac{ff'}{(f')^2+(g')^2}(\dot{u})^2 + \frac{f'f''+g'g''}{(f')^2+(g')^2}(\dot{v})^2 = 0. \end{cases}$$

Corollary

Any meridian is a geodesic. A parallel $\mathbf{X}(u, v_0)$ *is a geodesic if and only if* $f'(v_0) = 0$.

General geodesics

To study the behavior of general geodesics, we begin with the following lemma:

Lemma

Let $a_1(t), a_2(t)$ be smooth functions on $(T_1, T_2) \subset \mathbb{R}$ such that $a_1^2 + a_2^2 = 1$. For any $t_0 \in (T_1, T_2)$ and θ_0 such that $a_1(t_0) = \cos\theta_0$, $a_2(t_0) = \sin\theta_0$, there exists unique a smooth function $\theta(t)$ with $\theta(t_0) = \theta_0$ such that $a_1(t) = \cos\theta(t)$ and $a_2(t) = \sin\theta(t)$.

Proof of the lemma

Proof: Suppose θ satisfies the condition. Then $a_1' = -\theta' \sin\theta$, $a_2' = \theta' \cos\theta$. Hence $\theta' = a_1 a_2' - a_2 a_1'$. From this we have uniqueness. To prove existnce, fix $t_0 \in (T_1, T_2)$ and let θ_0 be such that $\cos\theta_0 = a_1(0)$, $\sin\theta_0 = a_2(0)$. Let

$$\theta(t) = \theta_0 + \int_{t_0}^{t} (a_2' a_1 - a_1' a_2) d\tau.$$

Let $f = (a_1 - b_1)^2 + (a_2 - b_2)^2$, where $b_1 = \cos\theta, b_2 = \sin\theta$. Then $f = 2 - 2a_1 b_1 - 2a_2 b_2$.

Proof of lemma, cont.

Then

$$\begin{aligned}
-\frac{1}{2}f' =& a_1' b_1 + a_1 b_1' + a_2' b_2 + a_2 b_2' \\
=& a_1' b_1 - \theta' a_1 b_2 + a_2' b_2 + \theta' a_2 b_1 \\
=& (a_2' a_1 - a_1' a_2)(-a_1 b_2 + a_2 b_1) + a_1' b_1 + a_2' b_2 \\
=& - a_1^2 a_2' b_2 + a_2 a_2' a_1 b_1 + a_1 a_1' a_2 b_2 - a_2^2 a_1' b_1 + a_1' b_1 + a_2' b_2 \\
=& - a_1^2 a_2' b_2 - a_1 a_1' a_1 b_1 - a_2 a_2' a_2 b_2 - a_2^2 a_1' b_1 + a_1' b_1 + a_2' b_2 \\
=& 0
\end{aligned}$$

because $a_1^2 + a_2^2 = 1$ and $a_1 a_1' + a_2 a_2' = 0$.

General geodesics, cont.

Now let $\alpha(s) = \mathbf{X}(u(s), v(s))$ be a geodesic on M parametrized by arc length. Let $\mathbf{e_1} = \mathbf{X}_u/|\mathbf{X}_u|$ and $\mathbf{e_2} = \mathbf{X}_v/|\mathbf{X}_v|$. Then $\mathbf{e}_1, \mathbf{e}_2$ are orthonormal. Let

$$\alpha' = a_1\mathbf{e_1} + a_2\mathbf{e_2}.$$

By the lemma there exists smooth function $\theta(s)$ such that $a_1 = \sin\theta$, $a_2 = \cos\theta$. Note that θ *is the angle between* α' *and the meridian*. That is:

$$\sin\theta = \langle \alpha', \mathbf{e_1} \rangle = f\dot{u}.$$

Clairaut's Theorem

Proposition (Clairaut's Theorem)

$r(s)\sin\theta(s)$ is constant along α, where $r(s)$ is the distance of $\alpha(s)$ from the z-axis.

Proof.

Denote the $\frac{d\alpha}{ds}$ by α' etc. Since $r(s) = f(v(s))$,

$$r' = f_v v'.$$

Also $\sin\theta = \langle \alpha', \mathbf{e_1} \rangle = u' f$, so $(\sin\theta)' = u'' f + u' v' f_v$.

$$\begin{aligned}(r\sin\theta)' =& f_v v' u' f + u'' f + f_v u' v' \\ =& f\left(u'' + \frac{2f_v}{f} u' v'\right) = 0.\end{aligned}$$

Another proof

Clairaut's Theorem revisited: In this case for the energy functional,

$$\mathcal{L} = \frac{1}{2}(f^2(v)(\dot{u})^2 + (f_v^2 + g_v^2)(\dot{v})^2).$$

Since geodesics satisfy the E-L equations, and

$$\frac{\partial}{\partial u}\mathcal{L} = 0,$$

and

$$\frac{\partial}{\partial \dot{u}}\mathcal{L} = f^2\dot{u},$$

hence we have

$$\frac{d}{dt}(f^2\dot{u}) = 0$$

along the geodesic.

Note that

$$\sin\theta = \langle \alpha', \mathbf{e}_1 \rangle = \langle \mathbf{X}_u \dot{u} + \mathbf{X}_v \dot{v}, \frac{\mathbf{X}_u}{|\mathbf{X}_u|} \rangle = f\dot{u}.$$

So $r(s)\sin\theta(s) = f(\alpha(s))\sin\theta(s) = f^2\dot{u}$.

Geodesics of surfaces of revolution, cont.

Let us analyse a geodesic $\alpha(s)$, $0 \leq s < L \leq \infty$, on the surface of revolution parametrized by arc length. Let us assume that

$g(v)$ is increasing, i.e. $g_v > 0$.

Let $r(s)$ and $\theta(s)$ be as in Clairaut's Theorem. Let $\theta_0 = \theta(0)$. We may assume that

$0 \leq \theta_0 \leq \frac{\pi}{2}$.

By Clairaut's Theorem,

$r(s) \sin \theta(s) = C$ for some constant $C \geq 0$.

Note that $r(s) \geq C$.
Case 1: If $\theta_0 = 0$, then $R = 0$ and it is a meridian.
Case 2: If $\theta = \pi/2$, then $r(0) = C$. If $f_v(\alpha(0)) = 0$, then it is a geodesic. If $f_v(\alpha(0)) \neq 0$, then near $s = 0$ and $s \neq 0$, α will stay in the region with $r(s) = f(\alpha(s)) > C$.
Case 3: Suppose $0 < \theta_0 < \pi/2$. Then α is going up initially. Moreover, near $s = 0$, $r(s) = C/\sin\theta(s) > C$. We consider two cases:

Case 3(i) There is no parallel above $\alpha(0)$ with radius C. That is all parallels above $\alpha(0)$ has radius larger than C. Then α will go up all the way.
Case 3(ii) There is a parallel above $\alpha(0)$ so that the radius is C. Let c be the first one above $\alpha(0)$. Then we have two more subcases:
(ii)(a) c is a geodesic. Then α will approach to C but never intersect C.
(ii)(b) c is not a geodesic, then α will touch C and bounces away.

To summarize, in the above settings, we have:

Proposition

(i) *If $C = 0$, then α is a meridian.*

(ii) *$R > 0$. Then geodesic will go up for all s, as long as $r > C$, i.e. the z coordinate of α is increasing in s. Either α does not come close to any parallel of radius C, and α will go up for all s, or α will be close to a parallel c of radius C. Let c be the first such parallel above α. Then we have the following cases:*

  (a) *c is a geodesic. Then α will not meet C and α will come arbitrarily close to C without intersecting C.*

  (b) *c is not a geodesic. Then there is $\alpha(s_0) \in c$ for some s_0 and α will bounce off from C and will turn downward.*