Isometry

Definition

Let $F : M_1 \to M_2$ be a diffeomorphism. F is said to be an isometry if for any $p \in M_1$ and $q = F(p)$, the linear map $dF : M_1 \to M_2$ is an isometry as inner product spaces. If there is an isometry from M_1 onto M_2, then M_1 is said to be isometric to M_2. F is a local isometry is for any $p \in M_1$, there exists open sets $p \in U_1$ and $q = F(p) \in U_2$ such that $F : U_1 \to U_2$ is an isometry.

A property or quantity of surfaces is said to be intrinsic if it is invariant under isometries. In other words, it is intrinsic if it depends only on the first fundamental form. E.g., the length of a curve is an intrinsic quantity.

Examples

- Let M_1 be the xy-plane parametrized by $\mathbf{X}(u,v) = (u,v,0)$. Let M_2 be the circular cylinder parametrized by $\mathbf{Y}(u,v) = (\cos u, \sin u, v)$. Consider the map $F : M_1 \to M_2$ so that $\mathbf{X}(u,v)$ is mapped into $\mathbf{Y}(u,v)$. This is not a diffeomorpism, but is a local diffeomorphism. Note that

$$dF(\mathbf{X}_u) = \mathbf{Y}_u, dF(\mathbf{X}_v) = \mathbf{Y}_v.$$

  Moreover, $\langle \mathbf{X}_u, \mathbf{X}_u \rangle = 1 = \langle \mathbf{Y}_u, \mathbf{Y}_u \rangle$, $\langle \mathbf{X}_v, \mathbf{X}_v \rangle = 1 = \langle \mathbf{Y}_v, \mathbf{Y}_v \rangle$, $\langle \mathbf{X}_u, \mathbf{X}_v \rangle = 0 = \langle \mathbf{Y}_u, \mathbf{Y}_u \rangle$. So this is a local isometry.

Examples, cont.

- Let M_1 be the xy-plane with the negative axis deleted, parametrized by $\mathbf{X}(\rho, \theta) = (\rho\cos\theta, \rho\sin\theta, 0)$. Let M_2 be the cone $\{z = k\sqrt{x^2 + y^2}\}$, so that $\cot\alpha = k$, $0 < 2\alpha < \pi$ is the angle at the vertex. Parametrize the cone by

$$\mathbf{Y}(\rho, \theta) = (\rho\sin\alpha\cos(\frac{\theta}{\sin\alpha}), \rho\sin\alpha\sin(\frac{\theta}{\sin\alpha}), \rho\cos\alpha)$$

Then it is a local isometry.

Examples, cont.

- Let M_1 be the catenoid parametrized by

$$\mathbf{X}(u, v) = (a \cosh v \cos u, a \cosh v \sin u, av)$$

Let M_2 be the helicoid given by

$$\mathbf{Y}(s, t) = (t \cos s, t \sin s, as).$$

Define a map F from M_1 to M_2 so that $(u, v) \to (s, t) = (u, a \sinh v)$. The Jacobian matrix is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & a \cosh v \end{pmatrix}$$

Then $dF(\mathbf{X}_u) = \mathbf{Y}_s, dF(\mathbf{X}_v) = a \cosh v \mathbf{Y}_t$. So

$$\langle \mathbf{X}_u, \mathbf{X}_u \rangle = a^2 \cosh^2 v = \langle dF(\mathbf{X}_u), dF(\mathbf{X}_u) \rangle$$

etc

Theorema Egregium of Gauss

Theorem

(Theorema Egregium of Gauss) The Guassian curvature K is invariant under isometries. That is to say, the Gaussian curvature depends only on the first fundamental form.

First proof

Recall the following.

- Let $A = (a_{ij})$, $B = (b_{ij})$ be two 3×3 matrices. Let $\mathbf{a}_i$ be the row vectors of A and $\mathbf{b}_j$ be the column vectors of B. Then

  $$AB = (\langle \mathbf{a}_i, \mathbf{b}_i \rangle).$$

  Hence $\det(AB) = \det(\langle \mathbf{a}_i, \mathbf{b}_i \rangle)$
  The above formula is still true if $\mathbf{b}_j$ are also row vectors of B because $\det(AB) = \det(AB^t)$.

- $$\det \begin{pmatrix} a & b & c \\ d & e & f \\ p & q & r \end{pmatrix} = \begin{pmatrix} 0 & b & c \\ d & e & f \\ p & q & r \end{pmatrix} + \begin{pmatrix} a & 0 & 0 \\ 0 & e & f \\ 0 & q & r \end{pmatrix}.$$

First proof

Proof:

- Let $\mathbf{X}(u^1, u^2)$ be a local parametrization of a regular surface, and let g_{ij} be the coefficients of the first fundamental form and h_{ij} be the second fundamental form.
- In the following, if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three vectors, $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ is the ordered triple product of the three vectors. This is just equal to $\det(\mathbf{a}, \mathbf{b}, \mathbf{c})$ as row vectors or as column vectors.
- $$h_{ij} = \langle \mathbf{N}, \mathbf{X}_{ij} \rangle = \frac{(\mathbf{X}_{ij}, \mathbf{X}_1, \mathbf{X}_2)}{\sqrt{\det(g_{ij})}} =: \frac{\Theta_{ij}}{\sqrt{\det(g_{ij})}}.$$
- So $K = \frac{\det(h_{ij})}{\det(g_{ij})} = \det(g_{ij})^{-2} \left(\Theta_{11}\Theta_{22} - \Theta_{12}^2\right)$

First proof, cont.

$$\Theta_{ij} = \det\left[\mathbf{X}_{ij}; \mathbf{X}_1; \mathbf{X}_2\right].$$

implies

$$\begin{aligned}
\Theta_{11}\Theta_{22} =& \det \begin{pmatrix} \langle \mathbf{X}_{11}, \mathbf{X}_{22}\rangle & \langle \mathbf{X}_{11}, \mathbf{X}_{1}\rangle & \langle \mathbf{X}_{11}, \mathbf{X}_{2}\rangle \\ \langle \mathbf{X}_{1}, \mathbf{X}_{22}\rangle & \langle \mathbf{X}_{1}, \mathbf{X}_{1}\rangle & \langle \mathbf{X}_{1}, \mathbf{X}_{2}\rangle \\ \langle \mathbf{X}_{2}, \mathbf{X}_{22}\rangle & \langle \mathbf{X}_{2}, \mathbf{X}_{1}\rangle & \langle \mathbf{X}_{2}, \mathbf{X}_{2}\rangle \end{pmatrix} \\
=& \det \begin{pmatrix} \langle \mathbf{X}_{11}, \mathbf{X}_{22}\rangle & \frac{1}{2}(g_{11})_1 & (g_{12})_1 - \frac{1}{2}(g_{11})_2 \\ (g_{12})_2 - \frac{1}{2}(g_{22})_1 & g_{11} & g_{12} \\ \frac{1}{2}(g_{22})_2 & g_{12} & g_{22} \end{pmatrix} \\
=& \det \begin{pmatrix} 0 & \frac{1}{2}(g_{11})_1 & (g_{12})_1 - \frac{1}{2}(g_{11})_2 \\ (g_{12})_2 - \frac{1}{2}(g_{22})_1 & g_{11} & g_{12} \\ \frac{1}{2}(g_{22})_2 & g_{12} & g_{22} \end{pmatrix} \\
&+ \langle \mathbf{X}_{11}, \mathbf{X}_{22}\rangle \det(g_{ij})
\end{aligned}$$

$$\begin{aligned}
\Theta_{12}^2 =& \det \begin{pmatrix} \langle \mathbf{X}_{12}, \mathbf{X}_{12} \rangle & \langle \mathbf{X}_{12}, \mathbf{X}_{1} \rangle & \langle \mathbf{X}_{12}, \mathbf{X}_{2} \rangle \\ \langle \mathbf{X}_{1}, \mathbf{X}_{12} \rangle & \langle \mathbf{X}_{1}, \mathbf{X}_{1} \rangle & \langle \mathbf{X}_{1}, \mathbf{X}_{2} \rangle \\ \langle \mathbf{X}_{2}, \mathbf{X}_{12} \rangle & \langle \mathbf{X}_{2}, \mathbf{X}_{1} \rangle & \langle \mathbf{X}_{2}, \mathbf{X}_{2} \rangle \end{pmatrix} \\
=& \det \begin{pmatrix} \langle \mathbf{X}_{12}, \mathbf{X}_{12} \rangle & \frac{1}{2}(g_{11})_2 & \frac{1}{2}(g_{22})_1 \\ \frac{1}{2}(g_{11})_2 & g_{11} & g_{12} \\ \frac{1}{2}(g_{22})_1 & g_{12} & g_{22} \end{pmatrix} \\
=& \det \begin{pmatrix} 0 & \frac{1}{2}(g_{11})_2 & \frac{1}{2}(g_{22})_1 \\ \frac{1}{2}(g_{11})_2 & g_{11} & g_{12} \\ \frac{1}{2}(g_{22})_1 & g_{12} & g_{22} \end{pmatrix} + \langle \mathbf{X}_{12}, \mathbf{X}_{12} \rangle \det(g_{ij})
\end{aligned}$$

First proof, cont.

Now

$$\begin{aligned}\langle \mathbf{X}_{11}, \mathbf{X}_{22}\rangle - \langle \mathbf{X}_{12}, \mathbf{X}_{12}\rangle \\ &= \langle \mathbf{X}_{1}, \mathbf{X}_{22}\rangle_1 - \underline{\langle \mathbf{X}_{1}, \mathbf{X}_{221}\rangle} - \langle \mathbf{X}_{1}, \mathbf{X}_{12}\rangle_2 + \underline{\langle \mathbf{X}_{1}, \mathbf{X}_{122}\rangle} \\ &= \left((g_{12})_2 - \frac{1}{2}(g_{22})_1 \right)_1 - \frac{1}{2} g_{11,22} \\ &= g_{12,12} - \frac{1}{2}(g_{11,22} + g_{22,11}).\end{aligned}$$

because
$\langle \mathbf{X}_1, \mathbf{X}_{22}\rangle = \langle \mathbf{X}_1, \mathbf{X}_2\rangle_2 - \langle \mathbf{X}_{12}, \mathbf{X}_2\rangle = \langle \mathbf{X}_1, \mathbf{X}_2\rangle_2 - \frac{1}{2}\langle \mathbf{X}_2, \mathbf{X}_2\rangle_1$

First proof, cont.

$$
\begin{aligned}
(\det(g_{ij}))^2 K = &\left(g_{12,12} - \frac{1}{2}(g_{11,22} + g_{22,11}) \right) \det(g_{ij}) \\
&+ \det \begin{pmatrix} 0 & \frac{1}{2}(g_{11})_1 & (g_{12})_1 - \frac{1}{2}(g_{11})_2 \\ (g_{12})_2 - \frac{1}{2}(g_{22})_1 & g_{11} & g_{12} \\ \frac{1}{2}(g_{22})_2 & g_{12} & g_{22} \end{pmatrix} \\
&- \det \begin{pmatrix} 0 & \frac{1}{2}(g_{11})_2 & \frac{1}{2}(g_{22})_1 \\ \frac{1}{2}(g_{11})_2 & g_{11} & g_{12} \\ \frac{1}{2}(g_{22})_1 & g_{12} & g_{22} \end{pmatrix}
\end{aligned}
$$

Hence K depends only on g_{ij} and their derivatives up to second order.

Christoffel symbols

Let $\mathbf{X}(u^1, u^2)$ is a coordinate parametrization. Let $\mathbf{X}_i = \mathbf{X}_{u^i}$, $g_{ij} = \langle \mathbf{X}_i, \mathbf{X}_j \rangle$, $(g^{ij}) = (g_{ij})^{-1}$. Then

$$\mathbf{X}_{ij} = \Gamma_{ij}^k \mathbf{X}_k + h_{ij}\mathbf{N}. \tag{1}$$

(Einstein summation convention: repeated indices mean summation.)

Γ_{ij}^k are called the Christoffel symbols for this parametrization.

To compute Γ^k_{ij}

Lemma

(i) $\Gamma^k_{ij} = \Gamma^k_{ji}$; *and*

(ii) $\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^2 g^{kl}\left(g_{il,j} + g_{jl,i} - g_{ij,l}\right).$ *where* $g_{ij,l} = \frac{\partial}{\partial u^l} g_{ij}$ *etc.*

Proof: $\mathbf{X}_{ij} = \mathbf{X}_{ji}$, so $\Gamma^k_{ij} = \Gamma^k_{ji}$. Note $\langle \mathbf{X}_{ij}, \mathbf{X}_l\rangle = \Gamma^k_{ij} g_{kl}$.

$$\begin{aligned} g_{il,j} =& \frac{\partial}{\partial u^j}\langle \mathbf{X}_i, \mathbf{X}_l\rangle \\ =& \langle \mathbf{X}_{ij}, \mathbf{X}_l\rangle + \langle \mathbf{X}_i, \mathbf{X}_{lj}\rangle \\ =& \langle \Gamma^k_{ij}\mathbf{X}_k, \mathbf{X}_l\rangle + \langle \mathbf{X}_i, \Gamma^k_{lj}\mathbf{X}_k\rangle \\ =& \Gamma^k_{ij} g_{kl} + \Gamma^k_{lj} g_{ki}. \end{aligned}$$

Proof, cont.

Hence we have

$$\begin{cases} g_{il,j} = & \Gamma^k_{ij} g_{kl} + \underline{\Gamma^k_{lj} g_{ki}}. \\ g_{jl,i} = & \Gamma^k_{ji} g_{kl} + \underline{\underline{\Gamma^k_{li} g_{kj}}}. \\ g_{ij,l} = & \underline{\underline{\Gamma^k_{il} g_{kj}}} + \underline{\Gamma^k_{jl} g_{ki}}. \end{cases}$$

(1)+(2)−(3):

$$g_{il,j} + g_{jl,i} - g_{ij,l} = 2\Gamma^k_{ij} g_{kl}.$$

From this the result follows.

Examples

- Let M be the xy-plane parametrized by $\mathbf{X}(u,v) = (u,v,0)$. Then $\Gamma_{ij}^k = 0$ for all i,j,k.
- If we use polar coordinates, $\mathbf{X}(r,\theta) = (r\cos\theta, r\sin\theta, 0)$. If $u^1 \leftrightarrow r, u^2 \leftrightarrow \theta$. Then $g_{11} = 1, g_{12} = 0, g_{22} = r^2$. So $g^{11} = 1, g^{12} = 0, g^{22} = r^{-2}$. Then

$$\Gamma_{ij}^1 = \frac{1}{2} g^{1k} \left(g_{ik,j} + g_{jk,i} - g_{ij,k}\right) = \frac{1}{2} \left(g_{i1,j} + g_{j1,i} - g_{ij,1}\right)$$

  Similarly,

$$\Gamma_{ij}^2 = \frac{1}{2} g^{2k} \left(g_{ik,j} + g_{jk,i} - g_{ij,k}\right) = \frac{1}{2} r^{-2} \left(g_{i2,j} + g_{j2,i} - g_{ij,2}\right).$$

So $\Gamma_{22}^1 = -r$, $\Gamma_{12}^2 = r^{-1}$, all other Γ's are zero.

Examples, cont.

Consider the surface of revolution given by

$$\mathbf{X}(u,v) = (\alpha(v)\cos u, \alpha(v)\sin u, \beta(v))$$

with $\alpha > 0$. Consider $u^1 \leftrightarrow u, u^2 \leftrightarrow v$. Then $g_{11} = \alpha^2, g_{12} = 0, g_{22} = (\alpha')^2 + (\beta')^2$. So $g^{11} = \alpha^{-2}, g^{12} = 0, g^{22} = \left((\alpha')^2 + (\beta')^2\right)^{-1}$.

$$\Gamma_{ij}^1 = \frac{1}{2} g^{1k}\left(g_{ik,j} + g_{jk,i} - g_{ij,k}\right) = \frac{1}{2}\alpha^{-2}\left(g_{i1,j} + g_{j1,i} - g_{ij,1}\right).$$

So

$$\Gamma_{11}^1 = \frac{1}{2}\alpha^{-2} g_{11,1} = 0,\ \Gamma_{22}^1 = \frac{1}{2}\alpha^{-2} g_{22,1} = 0,$$

$$\Gamma_{12}^1 = \frac{1}{2}\alpha^{-2} g_{11,2} = \frac{\alpha'}{\alpha}.$$

Examples, cont.

Similarly,

$$\Gamma_{ij}^2 = \frac{1}{2}g^{2k}\left(g_{ik,j} + g_{jk,i} - g_{ij,k}\right) = \frac{1}{2}g^{22}\left(g_{i2,j} + g_{j2,i} - g_{ij,2}\right).$$

Hence

$$\Gamma_{11}^2 = -\frac{1}{2}g^{22}g_{11,2} = -\frac{\alpha\alpha'}{(\alpha')^2 + (\beta')^2}, \Gamma_{22}^2 = \frac{1}{2}g^{22}g_{22,2} = \frac{\alpha'\alpha'' + \beta'\beta''}{(\alpha')^2 + (\beta')^2}.$$

$$\Gamma_{12}^2 = \frac{1}{2}g^{22}g_{22,1} = 0.$$

Examples, cont.

In general, if $g_{12} = 0$, then

$$\Gamma_{ij}^{k} = \frac{1}{2}\sum_{l=1}^{2} g^{kl}\left(g_{il,j} + g_{jl,i} - g_{ij,l}\right) = \frac{1}{2}g^{kk}\left(g_{ik,j} + g_{jk,i} - g_{ij,k}\right)$$

no summation. So

$$\Gamma_{11}^{1} = \frac{1}{2}g^{11}g_{11,1},\ \Gamma_{11}^{2} = -\frac{1}{2}g^{22}g_{11,2};$$

$$\Gamma_{22}^{1} = -\frac{1}{2}g^{11}g_{22,1},\ \ \Gamma_{22}^{2} = \frac{1}{2}g^{22}g_{22,2};$$

$$\Gamma_{12}^{1} = \frac{1}{2}g^{11}g_{11,2},\ \ \Gamma_{12}^{2} = \frac{1}{2}g^{22}g_{22,1}.$$

Second proof of Theorema Egregium of Gauss

Theorem

With the above notations, then

$$2K = g^{ij}\left(\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^k_{lk}\Gamma^l_{ji} - \Gamma^k_{lj}\Gamma^l_{ki}\right) = g^{ij}(\Gamma^k_{i[j,k]} + \Gamma^k_{l[k}\Gamma^l_{j]i}).$$

Here $T_{[ij]k} = T_{ijk} - T_{jik}$ *etc.*

Compare with Riemannian curvature tensor inhigher dimensions:

$$R^l_{ijk} = \Gamma^l_{ik,j} - \Gamma^l_{ij,k} + \Gamma^l_{js}\Gamma^s_{ik} - \Gamma^l_{ks}\Gamma^s_{ij}$$

Proof

Proof: Let $\mathcal{S}$ be the shape operator, then

$$-\mathbf{N}_i = \mathcal{S}(\mathbf{X}_i) = a_i^j \mathbf{X}_j.$$

$$\begin{aligned}\mathbf{X}_{ijm} =& h_{ij,m}\mathbf{N} + h_{ij}\mathbf{N}_m + \Gamma_{ij,m}^k \mathbf{X}_k + \Gamma_{ij}^k \mathbf{X}_{km} \\ =& \left(h_{ij,m} + \Gamma_{ij}^k h_{km}\right)\mathbf{N} + \left(-h_{ij}a_m^k + \Gamma_{ij,m}^k + \Gamma_{ij}^s \Gamma_{sm}^k\right)\mathbf{X}_k\end{aligned}$$

Since $\mathbf{X}_{ijm} = \mathbf{X}_{imj}$, we have

$$\left(-h_{ij}a_m^k + \Gamma_{ij,m}^k + \Gamma_{ij}^s \Gamma_{sm}^k\right)\mathbf{X}_k = \left(-h_{im}a_j^k + \Gamma_{im,j}^k + \Gamma_{im}^s \Gamma_{sj}^k\right)\mathbf{X}_k$$

Or

$$h_{ij}a_m^k - h_{im}a_j^k = \Gamma_{ij,m}^k - \Gamma_{im,j}^k + \Gamma_{ij}^s \Gamma_{ms}^k - \Gamma_{im}^s \Gamma_{js}^k$$

Proof, cont.

Now the matrix of the shape operator is:

$$(a^i_j)^T = (h_{ij})(g_{ij})^{-1} \Rightarrow (g_{ij})(a^i_j) = (h_{ij})$$

So $h_{ji} = h_{ij} = g_{il}a^l_j$. Hence

$$g_{il}a^l_j a^k_m - g_{il}a^l_m a^k_j = \Gamma^k_{ij,m} - \Gamma^k_{im,j} + \Gamma^s_{ij}\Gamma^k_{ms} - \Gamma^s_{im}\Gamma^k_{js}.$$

Let $m = k$ and sum on k

$$\begin{aligned} & g^{ij}\left(\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^s_{ij}\Gamma^k_{ks} - \Gamma^s_{ik}\Gamma^k_{js}\Gamma_{ij,k}\right) \\ =& g^{ij}\left(g_{il}a^l_j a^k_k - g_{il}a^l_k a^k_j\right) \\ =& (\sum_k a^k_k)^2 - \sum_{l,k} a^k_l a^l_k \\ =& 2a_{11}a_{22} - 2a^2_1 a^1_2 \\ =& 2K \end{aligned}$$

Compatibility conditions

For your reference: Given (g_{ij}) which is symmetric and positive definite and (h_{ij}) which is symmetric, can we find $\mathbf{X}(u^1, u^2)$ so that the first fundamental form is h_{ij}? If $\mathbf{X}_i$ exist, then we can find $\mathbf{X}$. The restriction on $\mathbf{X}_i$ are

$$\mathbf{X}_{ijk} = \mathbf{X}_{ikj}, \quad \mathbf{N}_{ij} = \mathbf{N}_{ji}.$$

Hence we have

$$\left(-h_{ij} a_m^k + \Gamma_{ij,m}^k + \Gamma_{ij}^s \Gamma_{sm}^k\right) \mathbf{X}_k = \left(-h_{im} a_j^k + \Gamma_{im,j}^k + \Gamma_{im}^s \Gamma_{sj}^k\right) \mathbf{X}_k$$

with $g_{il} a_j^l = h_{ij}$. We have three relations for each $\mathbf{X}_i$. Now

$$\begin{aligned} -\mathbf{N}_{ij} =& (a_i^k \mathbf{X}_k)_j \\ =& (a_i^k)_j \mathbf{X}_k + a_i^k (\Gamma_{jk}^l \mathbf{X}_l + a_i^k h_{jk} \mathbf{N}) \\ =& \left((a_i^k)_j + a_i^l \Gamma_{jl}^k\right) \mathbf{X}_k + a_i^k h_{jk} \mathbf{N} \end{aligned}$$

So we also need, for $k = 1, 2$

$$(a_i^k)_j + a_i^l \Gamma_{jl}^k = (a_j^k)_i + a_j^l \Gamma_{il}^k.$$

These are called Gauss equations and Mainardi-Codazzi equations respectively.