THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4060 Complex Analysis 2022-23
Homework 3 solutions

1. (Exercise 1 in textbook) By the product formula of $\frac{1}{\Gamma(s)}$, we have:

$$\begin{aligned}
\frac{1}{\Gamma(s)} &= e^{\gamma s} s \prod_{n=1}^{\infty}(1+\frac{s}{n})e^{-\frac{s}{n}} \\
&= \lim_{N\to\infty} s e^{s(\sum_{n=1}^{N}\frac{1}{n}-logN)} \prod_{n=1}^{N}(\frac{s+n}{n})e^{-\frac{s}{n}} \\
&= \lim_{N\to\infty} s e^{s(\sum_{n=1}^{N}\frac{1}{n}-logN-\sum_{n=1}^{N}\frac{1}{n})} \prod_{n=1}^{N}(\frac{s+n}{n}) \\
&= \lim_{N\to\infty} s e^{s(-logN)} \prod_{n=1}^{N}(\frac{s+n}{n}) \\
&= \lim_{N\to\infty} s N^{-s} \prod_{n=1}^{N}(\frac{s+n}{n}) \\
&= \lim_{N\to\infty} (\frac{N^s N!}{s(s+1)\dots(s+N)})^{-1} \\
\Rightarrow \quad \Gamma(s) &= \lim_{N\to\infty} \frac{N^s N!}{s(s+1)\dots(s+N)} \ \forall s \neq \ 0,\ -1,\ -2\dots
\end{aligned}$$

2. (Exercise 5 in textbook) Recall that for $\Gamma(s)$, we have:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}$$

Let $s = \frac{1}{2} + it$,

$$\begin{aligned}
|\Gamma(\frac{1}{2}+it)\Gamma(\frac{1}{2}-it)| &= \frac{\pi}{|\sin(\pi(\frac{1}{2}+it))|} \\
&= \frac{2\pi}{|e^{(\frac{1}{2}+it)i\pi} - e^{-(\frac{1}{2}+it)i\pi}|} \\
&= \frac{2\pi}{e^{\pi t}+e^{-\pi t}}
\end{aligned}$$

On the other hand, by definition of Gamma function:

$$\begin{aligned}
\Gamma(\frac{1}{2}-it) &= \int_0^\infty e^{-u}u^{\frac{1}{2}-it-1}du \\
&= \int_0^\infty e^{-u}\overline{u^{\frac{1}{2}+it-1}}du \\
&= \overline{\int_0^\infty e^{-u}u^{\frac{1}{2}+it-1}du} \\
&= \overline{\Gamma(\frac{1}{2}+it)}
\end{aligned}$$

where $\overline{f(x)}$ denote the complex conjugation of $f(x)$. Thus we have:

$$|\Gamma(\frac{1}{2}+it)| = \sqrt{|\Gamma(\frac{1}{2}+it)\Gamma(\frac{1}{2}-it)|}$$
$$= \sqrt{\frac{2\pi}{e^{\pi t}+e^{-\pi t}}}$$

3. (Exercise 6 in textbook) The definition of Euler's constant is:

$$\gamma = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k} - \log(n)$$
$$= \lim_{n\to\infty}\sum_{k=1}^{n}(\frac{1}{k} - \int_k^{k+1}\frac{1}{x}dx)$$
$$= \lim_{n\to\infty}\sum_{k=1}^{2n}(\frac{1}{k} - \int_k^{k+1}\frac{1}{x}dx)$$

which converges, and similarly

$$\frac{\gamma}{2} = \lim_{n\to\infty}\sum_{k=1}^{n}(\frac{1}{2k} - \int_k^{k+1}\frac{1}{2x}dx)$$

Therefore,

$$\lim_{n\to\infty} 1+\frac{1}{3}+\frac{1}{5}+\dots\frac{1}{2n-1}-\frac{1}{2}\log n = \lim_{n\to\infty} 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{2n}-\log 2n$$
$$-(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\dots+\frac{1}{2n}-\frac{1}{2}\log n)$$
$$= \gamma - \frac{1}{2}\gamma + \log 2$$

4. (Exercise 17 in textbook)

   (a) We use the same way as in the proof that Gamma function is holomorphic in $s \in \{s|0 < Re(s)\}$. Since we assumed $f(x)$ decay faster than any polynomial.

   $$\int_M^\infty |f(x)x^{s-1}|dx = \int_M^\infty |f(x)x^{s-1+2}|\frac{1}{x^2}dx$$
   $$\le C_1\int_M^\infty \frac{1}{x^2}dx \to 0 \text{ as } M\to\infty \text{ when } 0<k_1<Re(s)<k_2\ \forall k_1,k_2$$

   And similarly,

   $$\int_0^\epsilon |f(x)x^{s-1}|dx = \int_0^\epsilon |f(x)|x^{Re(s)-1}dx$$
   $$\le C_2\int_0^\epsilon x^{Re(s)-1}dx \to 0 \text{ as } \epsilon\to 0\quad \forall s \text{ with } 0<k_1<Re(s)<k_2\ \forall k_1,k_2$$

On the other hand, $f(x)x^{s-1}$ is holomorphic in $s \in \{s|0 < k_1 < Re(s) < k_2\}$ and continuous in $(s, x) \in \{s|0 < k_1 < Re(s) < k_2\} \times [\epsilon, M]$. By theorem 5.4 of Chap 2 in textbook,

$$F_N(x) = \int_{\frac{1}{N}}^{N} f(x)x^{s-1}dx$$

is holomorphic in $s \in \{s|0 < k_1 < Re(s) < k_2\}$. Therefore,

$$|\int_0^\infty f(x)x^{s-1}dx - \int_{\frac{1}{N}}^{N} f(x)x^{s-1}dx| < \int_0^{\frac{1}{N}} |f(x)x^{s-1}|dx + \int_N^\infty |f(x)x^{s-1}|dx$$
$$\to 0 \text{ as } N \to \infty$$

Thus for any $0 < k_1 < k_2$ we have a sequence of holomorphic function $f_N(s)$ uniformly converge to $\int_0^\infty f(x)x^{s-1}dx$. Thus $\int_0^\infty f(x)x^{s-1}dx$ is also holomorphic in $s \in \{s|0 < k_1 < Re(s) < k_2\}$. Hence holomorphic in $s \in \{s|0 < Re(s)\}$. Since $\frac{1}{\Gamma(s)}$ is an entire function and k_1, k_2 arbitrary, we have $I(s)$ is holomorphic in $s \in \{s|0 < Re(s)\}$.

Then we consider the integration by parts in hint:

$$\begin{aligned}
\frac{(-1)^k}{\Gamma(s+k)} \int_0^\infty f^{(k)}(x)x^{s+k-1}dx &= \frac{(-1)^k}{\Gamma(s+k)}(f^{(k-1)}x^{s+k-1}|_0^\infty \\
&- \int_0^\infty (s+k-1)f^{(k-1)}(x)x^{s+k-2}dx) \\
&= \frac{(-1)^{k-1}}{\Gamma(s+k-1)}(\int_0^\infty f^{(k-1)}(x)x^{s+k-2}dx) \\
&= \dots \\
&= \frac{1}{\Gamma(s)}(\int_0^\infty f^{(0)}(x)x^{s-1}dx) \\
&= \frac{1}{\Gamma(s)}(\int_0^\infty f(x)x^{s-1}dx) = I(s)
\end{aligned}$$

$\lim_{x\to\infty} f^{(k-1)}x^{s+k-1} = 0$ is again because we assumed $f(x)$ decay faster than any polynomial. But

$$\frac{(-1)^k}{\Gamma(s+k)} \int_0^\infty f^{(k)}(x)x^{s+k-1}dx$$

is holomorphic in $s \in \{s| -k < Re(s)\}$ by the similar argument. Let $k \to \infty$, by uniqueness of holomorphic function, $I(s)$ has an analytic continuation as an entire function.

(b) By part (a),

$$\begin{aligned}
\frac{1}{\Gamma(s)}(\int_0^\infty f(x)x^{s-1}dx) &= \frac{-1}{\Gamma(s+1)}(\int_0^\infty f(x)^{(1)}x^{s+1-1}dx) \\
&= \frac{(-1)^{n+1}}{\Gamma(s+n+1)}(\int_0^\infty f(x)^{(n+1)}x^{s+n+1-1}dx)
\end{aligned}$$

Let $s = 0$ in the first equation, we get:

$$I(0) = -f(x)|_0^\infty = f(0)$$

Let $s = n$ in the second equation, we get:

$$I(-n) = (-1)^n f(x)^{(n)}|_0^\infty = (-1)^n f^{(n)}(0).$$