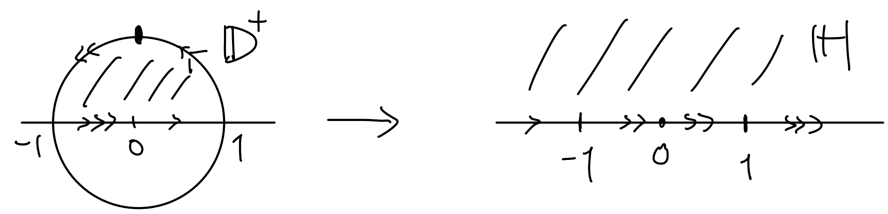
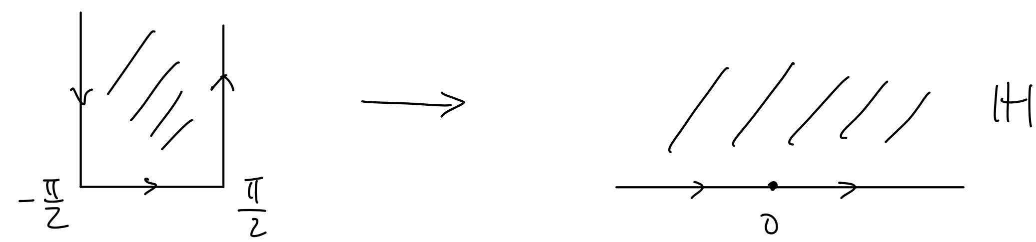
Eg 7 (Ex 5) $f(z) = -\frac{1}{2}\left(z+\frac{1}{z}\right)$ maps conformally



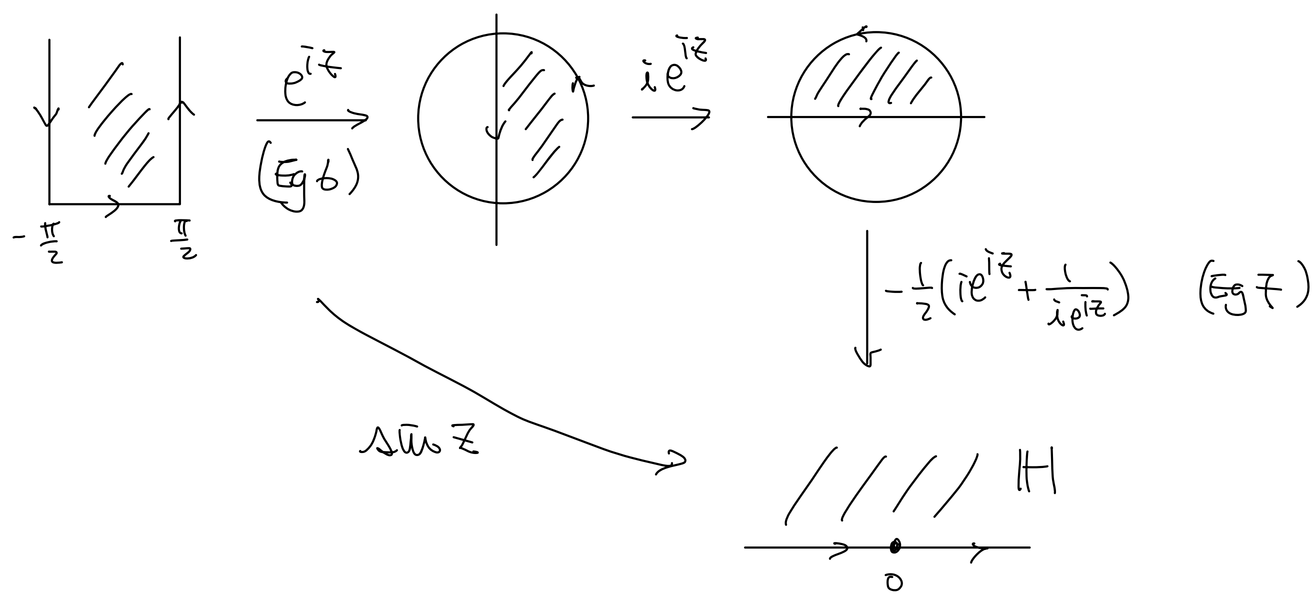
(misprint in the textbook concerning the boundary behaviour, confused with $z \mapsto \frac{1}{2}\left(z+\frac{1}{z}\right)$)

Eg 8 $f(z) = \sin z$ maps conformally (misprint in Textbook, confused domain and target)



Note $f(z) = \sin z = \dfrac{e^{iz}-e^{-iz}}{2i} = -\frac{1}{2}\left(\dfrac{-e^{iz}}{i} + \dfrac{e^{-iz}}{i}\right)$

$= -\frac{1}{2}\left(ie^{iz} + \dfrac{1}{ie^{iz}}\right)$



1.2 The Dirichlet Problem in a Strip

Dirichlet Problem in the open set Ω consists of solving

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega \end{cases}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the Laplacian (operator)

f = given (continuous) function on $\partial\Omega$.

(i.e. Dirichlet problem = Boundary Value Problem for the Laplace equation)

Known Fact: Solution to Dirichlet Problem in the unit disk $\mathbb{D}$.

Recall: using polar coordinates

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

Let f be a continuous function on $\partial\mathbb{D} = S^1$.

Then f can be represented as a (periodic) function of θ

$$f(\theta), \quad 0 \leq \theta \leq 2\pi.$$

Then the unique solution to $\begin{cases} \Delta u = 0 & \text{in } \mathbb{D} \\ u = f & \text{on } \partial\mathbb{D} = S^1 \end{cases}$

is given by $$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - \varphi) f(\varphi)\, d\varphi$$

where $$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2}$$

(See Textbook for reference)

In this section, we illustrate how to use conformal maps and the solution of Dirichlet problem in the unit disc to solve Dirichlet Problem in a more general domain Ω in $\mathbb{C}$.

Lemma 1.3 $F: U \to V$ holo. (U, V open in $\mathbb{C}$)

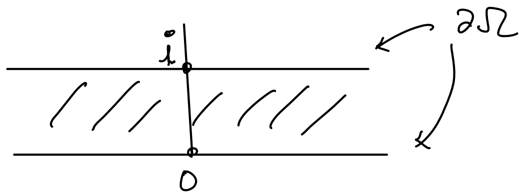
If $u: V \to \mathbb{C}$ is harmonic (i.e. $\Delta u = 0$),

then $u \circ F: U \to \mathbb{C}$ is harmonic.

Pf Easy exercise using Chain rule and Cauchy-Riemann equation.

(Or observing that $\exists$ holo. G on $\overline{U}$ s.t. $\mathrm{Re}\, G = u$.)

Dirichlet Problem in the strip $\Omega = \{x+iy : x \in \mathbb{R},\ 0 < y < 1\}$



Then boundary $\partial\Omega$ of Ω consists of two components

$$L_0 = \{x+iy : y = 0\} \quad \& \quad L_1 = \{x+iy : y = 1\}$$

Let $f_0 : L_0 \to \mathbb{R}$ and $f_1 : L_1 \to \mathbb{R}$ be continuous functions (and represented as functions of x only)

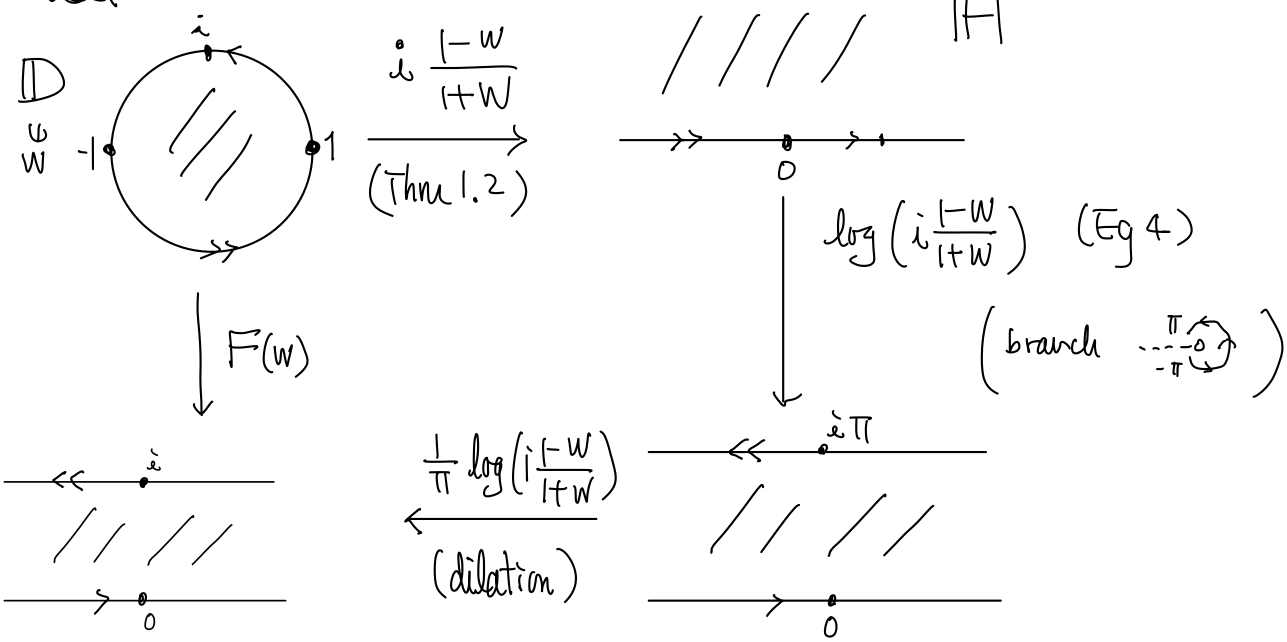
We need to find $u(x,y)$ such that $\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u(x,0) = f_0(x) \\ u(x,1) = f_1(x) \end{cases}$

For techincal reason, let consider special cases that

$$\lim_{|x|\to\infty} f_0(x) = \lim_{|x|\to\infty} f_1(x) = 0 .$$

<u>Step 1</u>: Find a conformal map $F: \mathbb{D} \to \Omega$.

Recall that



$\therefore$ $F(w) = \frac{1}{\pi}\log\left(i\,\frac{1-w}{1+w}\right)$ maps $\mathbb{D}$ conformally onto Ω.

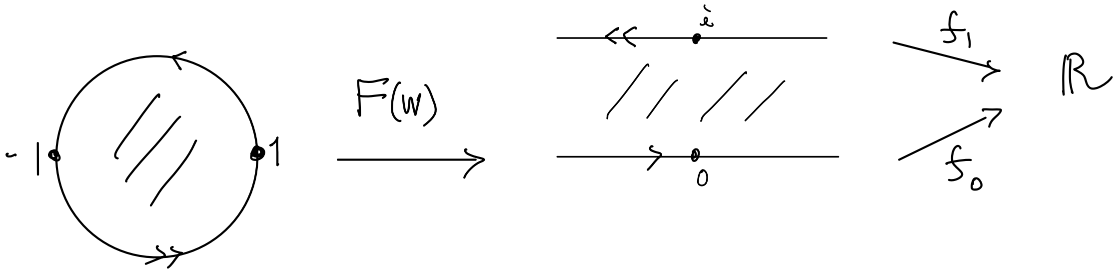
Easy calculation $\Rightarrow$

$$G(z) = \frac{i - e^{\pi z}}{i + e^{\pi z}} : \Omega \to \mathbb{D}$$

is the inverse $(G = F^{-1})$

Boundary behaviour : $\varphi : -\pi \to 0 \;\leftrightarrow\; F(e^{i\varphi}) : i+\infty \to i-\infty$

$\varphi : 0 \to \pi \;\leftrightarrow\; F(e^{i\varphi}) : -\infty \to +\infty$ (x-axis)



Define $\tilde{f} : S^1 = \partial\mathbb{D} \to \mathbb{R}$ by

$$\tilde{f}(\varphi) = \begin{cases} f_0(F(e^{i\varphi})) & , \quad 0 < \varphi < \pi \\ f_1(F(e^{i\varphi}) - i) & , \quad -\pi < \varphi < 0 \\ 0 & , \quad \varphi = 0, \pm\pi \end{cases}$$

Then by $\lim_{|x|\to\infty} f_0(x) = \lim_{|x|\to\infty} f_1(x) = 0$, $\tilde{f}$ is continuous.

Using the solution to the Dirichlet problem in the unit disc $\mathbb{D}$,

$$\tilde{u}(w) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - \varphi)\, \tilde{f}(\varphi)\, d\varphi$$

is a harmonic function in $\mathbb{D}$ with boundary value

$$\tilde{u}\big|_{\partial\mathbb{D}} = \tilde{f}.$$

Then Lemma 1.3 $\Rightarrow$ $u = \tilde{u} \circ G : \Omega \to \mathbb{R}\ (\subset \mathbb{C})$

is the solution to the Dirichlet problem in the strip Ω.

More explicitly, we have

$$\boxed{u(x,y) = \frac{\sin \pi y}{2} \left(\int_{-\infty}^{\infty} \frac{f_0(x-t)}{\cosh(\pi t) - \cos \pi y}\, dy + \int_{-\infty}^{\infty} \frac{f_1(x-t)}{\cosh(\pi t) + \cos \pi y}\, dy \right) \quad (0 < y < 1)}$$

(Details omitted see Ex 7 & discussion on page 216 in the Textbook)