2. Reduction to the functions ψ and ψ_1

Def Tchebychev's ψ-function

$$\psi(x) = \sum_{p^m \le x} \log p = \sum_{p \le x} \left[\frac{\log x}{\log p}\right] \log p$$

$$= \sum_{1 \le n \le x} \Lambda(n)$$

where $\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m \text{ for some prime } p \ \& \ m \ge 1 \\ 0, & \text{otherwise} \end{cases}$

Remarks: (i) The sum $\sum_{p^m \le x}$ is over those integers of the form p^m & $\le x$.

(ii) $[u]$ = greatest integer $\le u$.

Prop 2.1 If $\psi(x) \sim x$ as $x \to \infty$, then $\pi(x) \sim \frac{x}{\log x}$ as $x \to \infty$

Pf omitted as it is completely a "real" analysis argument.
(Reading Exercise)

Remark: Converse of Prop. 2.1 holds.

Def $\psi_1(x) = \int_1^x \psi(u)\,du$

Prop 2.2 If $\psi_1(x) \sim \frac{x^2}{2}$ as $x \to \infty$, then $\psi(x) \sim x$ as $x \to \infty$, and therefore $\pi(x) \sim \frac{x}{\log x}$ as $x \to \infty$.

Pf omitted as it is completely a "real" analysis argument.
(Reading Exercise)

Prop 2.3 $\forall c > 1$

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds \qquad (6)$$

(The integral is along the vertical line $\mathrm{Re}(s)=c$.)

Pf: Step 1:

$$\boxed{-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s} \qquad \mathrm{Re}\, s>1}$$

In Lemma 1.3, we have proved that

$$\log\zeta(s) = \sum_{p,m}\frac{1}{m}p^{-ms}$$

$$\Rightarrow \quad \frac{\zeta'(s)}{\zeta(s)} = \sum_{p,m}\frac{1}{m}(-m\log p)\,p^{-ms} = -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}$$

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Step 2:

Lemma 2.4 If $c>0$, then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{a^s}{s(1+s)}ds = \begin{cases} 0, & \text{if } 0<a\le 1 \\ 1-\frac{1}{a}, & \text{if } 1\le a\end{cases}$$

(The integral is along the vertical line $\mathrm{Re}(s)=c$.)

Clearly the integral converges as $|a^s| = a^c$.

Case 1 $a \geq 1$.

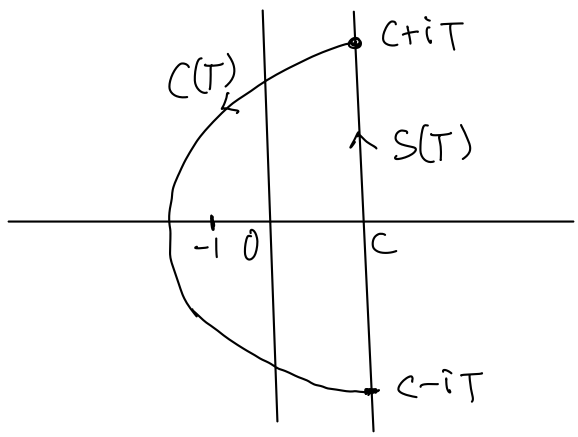
Let $\beta = \log a \geq 0$ and consider

$$f(s) = \frac{a^s}{s(s+1)} = \frac{e^{s\beta}}{s(s+1)}$$ which is meromophic

with simple poles at $s=0$ & $s=-1$ with

$$\operatorname{res}_{s=0} f = 1 \quad \text{and} \quad \operatorname{res}_{s=-1} f = -\frac{1}{a}$$

Let $\Gamma(T) = S(T) + C(T)$ be the contour as in the figure ($T > c+1$)



where $S(T)$ = vertical line segment from $c-iT$ to $c+iT$;

$C(T)$ = left half-circle of radius T centered at c.

Then Residue Theorem $\Rightarrow$ $$\frac{1}{2\pi i}\int_{\Gamma(T)} f(s)\, ds = 1 - \frac{1}{a}.$$

Now if $s = \sigma + it \in C(T)$, then $|s(s+1)| \geq (T-c)(T-c-1)$

$$\Rightarrow \left|\int_{C(T)} f(s)\, ds\right| = \left|\int_{C(T)} \frac{e^{\beta s}}{s(s+1)}\, ds\right| \leq \int_{C(T)} \frac{|e^{\beta s}|}{|s(s+1)|}\, ds$$

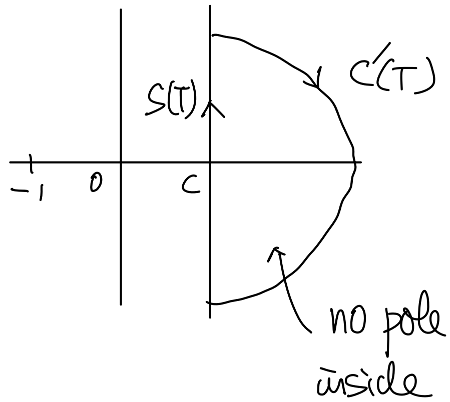
($\operatorname{Re} s \leq c$, $\beta \geq 0$) $$\leq \frac{e^{\beta c}}{(T-c)(T-c-1)} \cdot \pi T \longrightarrow 0 \text{ as } T \to \infty,$$

$\therefore \quad 2\pi i\left(1-\frac{1}{a}\right) = \int_{\Gamma(T)} f(s)ds = \int_{S(T)} f(s)ds + \int_{C(T)} f(s)ds$

$$\to \int_{c-i\infty}^{c+i\infty} f(s)ds \quad \text{as } T \to \infty .$$

$$\therefore \quad \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{a^s}{s(s+1)}ds = 1-\frac{1}{a} \quad \text{for } a \geq 1 .$$

<u>Case 2</u> $\quad 0 < a \leq 1$



Similar to case 1, we have

$$\left|\int_{C'(T)} f(s)ds\right| \leq \int_{C'(T)} \frac{|a^s|}{|s(s+1)|}ds$$

$$= \int_{C'(T)} \frac{|e^{-s\log\frac{1}{a}}|}{|s(s+1)|}ds \qquad \left(\log\frac{1}{a} > 0\right)$$

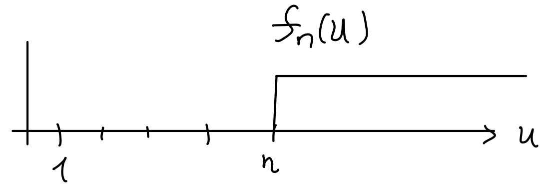
$$(\operatorname{Re} s \geq c) \qquad \leq e^{-c\log\frac{1}{a}} \cdot \frac{1}{(T+c)(T+c+1)} \cdot \pi T \to 0$$

and the same argument gives

$$0 = \int_{S(T)} f(s)ds + \int_{C(T)} f(s)ds$$

$$\to \int_{c-i\infty}^{c+i\infty} \frac{a^s}{s(s+1)}ds \quad \text{as } T \to \infty . \quad \#$$

Step 3 $\boxed{\psi_1(x) = \sum_{n\le x} \Lambda(n)(x-n)}$



$$\psi(u) = \sum_{1\le n\le u} \Lambda(n)$$

$$= \sum_{n=1}^{\infty} \Lambda(n) f_n(u) \qquad \text{where } f_n(u) = \begin{cases} 1 & \text{if } u \ge n \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad \psi_1(x) = \int_1^x \psi(u)\,du$$

$$= \int_0^x \psi(u)\,du \qquad \text{as } \psi(u) = 0 \text{ for } 0<u\le 1$$

$$= \int_0^x \sum_{n=1}^{\infty} \Lambda(n) f_n(u)\,du$$

$$= \sum_{n=1}^{\infty} \Lambda(n) \int_0^x f_n(u)\,du$$

$$= \sum_{n\le x} \Lambda(n) \int_0^x f_n(u)\,du \qquad \text{as } n > x \ge u \Rightarrow f_n(u) = 0$$

$$= \sum_{n\le x} \Lambda(n)(x-n) \quad \divideontimes$$

Final Step: For $c > 1$

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds$$

(by Step 1)

$\mathrm{Re}\, s = c > 1$

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}\right) ds$$

$$= x \cdot \sum_{n=1}^{\infty} \Lambda(n) \cdot \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\left(\frac{x}{n}\right)^s}{s(s+1)}\,ds$$

(by Step 2) $= x \sum_{n=1}^{\infty} \Lambda(n) \cdot \begin{cases} 1 - \frac{1}{(\frac{x}{n})}, & \text{if } \frac{x}{n} \geq 1 \\ 0, & \text{otherwise} \end{cases}$

$= x \sum_{n \leq x} \Lambda(n) \left(1 - \frac{n}{x}\right)$

$= \sum_{n \leq x} \Lambda(n)(x - n)$

(by Step 3) $= \psi_1(x)$.

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