**Thm 2.4** $\zeta(s)$ has a meromorphic continuation into the whole $\mathbb{C}$ whose only singularity is a simple pole at $s=1$.

Pf: By definition of $\xi(s)$, we have

$$\zeta(s) = \pi^{\frac{s}{2}} \frac{\xi(s)}{\Gamma(\frac{s}{2})}$$

By Thm 1.6, $1/\Gamma(\frac{s}{2})$ is entire with simple zeros at

$s = 0, -2, -4, \cdots$

$\Rightarrow$ $s=0$ is a removable singularity of $\xi(s)/\Gamma(\frac{s}{2})$.

$\Rightarrow$ $\zeta(s)$ is meromorphic with a simple pole at $s=1$ only. #

Question: What is $\operatorname{res}_{s=1} \zeta(s)$ ? (answer $=1$, Ex!)

**Prop 2.5** $\exists$ seq. of entire function $\{\delta_n(s)\}_{n=1}^{\infty}$ such that

- $|\delta_n(s)| \le \dfrac{|s|}{n^{\operatorname{Re}s+1}}$, $\forall s \in \mathbb{C}$ and

(8) —— • $\displaystyle\sum_{1\le n<N} \frac{1}{n^s} - \int_1^N \frac{dx}{x^s} = \sum_{1\le n<N} \delta_n(s)$, $(N = 2, 3, \cdots)$.

Pf: Define $\displaystyle\delta_n(s) = \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx$ (entire)

By path integral of $f(z) = s z^{-s-1}$ (for $\operatorname{Re} z > 0$)

along $z(t) = x + t(n-x)$, $t \in [0,1]$, we have

$$\left| \frac{1}{n^s} - \frac{1}{x^s} \right| \le \int_0^1 |s| \left| [x + t(n-x)]^{-s-1} \right| |n-x| \, dt$$

$$\le \frac{|s|}{n^{\sigma+1}} \qquad \text{for } x \in [n, n+1] \quad (\text{where } \sigma = \operatorname{Re} s)$$

$$\Rightarrow \quad |\delta_n(s)| \le \int_n^{n+1} \left| \frac{1}{n^s} - \frac{1}{x^s} \right| dx \le \frac{|s|}{n^{\sigma+1}} .$$

Summing up

$$\sum_{n=1}^{N-1} \delta_n(s) = \sum_{n=1}^{N-1} \int_n^{n+1} \left( \frac{1}{n^s} - \frac{1}{x^s} \right) dx = \sum_{n=1}^{N-1} \frac{1}{n^s} - \int_1^N \frac{dx}{x^s} . \quad \#$$

<u>Cor 2.6</u> For $\operatorname{Re}(s) > 0$,

$$\zeta(s) - \frac{1}{s-1} = H(s)$$

where $H(s) = \sum_{n=1}^{\infty} \delta_n(s)$ is holo. in $\{ \operatorname{Re}(s) > 0 \}$.

<u>Pf</u>: For $\operatorname{Re}(s) > 1$,

the LHS of the formula (8) $\rightarrow \zeta(s) - \frac{1}{s-1}$ as $N \to \infty$.

For the RHS, Prop 2.5 $\Rightarrow$ $|\delta_n(s)| \le \frac{|s|}{n^{\operatorname{Re} s + 1}}$, $\forall n$

$\Rightarrow$ $\sum_{n=1}^{\infty} \delta_n(s)$ converges uniformly on $\{ |s| < R \} \cap \{ \operatorname{Re}(s) > 0 \}$, $\forall R > 0$;
(locally)

$\left( \begin{array}{l} \text{in fact on } \{|s|<R\} \cap \{\mathrm{Re}(s) \geq \delta\}, \ \forall R>0 \ \& \ \delta>0 \\ \quad \text{since } \sum \frac{1}{n^{\delta+1}} < \infty \text{ for } \delta>0 \end{array} \right)$

Hence $H(s) = \sum_{n=1}^{\infty} \delta_n(s)$ is holo. on $\{\mathrm{Re}(s)>0\} \supset \{\mathrm{Re}(s)>1\}$

$\therefore \qquad \zeta(s) - \frac{1}{s-1} = H(s)$ for $\mathrm{Re}(s)>1$.

Note that $\zeta(s)$ has analytic continuation to $\{\mathrm{Re}(s)>0\} \setminus \{s=1\}$

and $\mathrm{res}_{s=1} \zeta(s) = 1$. By uniqueness, the equality

$$\zeta(s) - \frac{1}{s-1} = H(s)$$

also holds on $\{\mathrm{Re}(s)>0\}$. $\#$

**Prop 2.7** Suppose $s = \sigma + it$, $(\sigma, t \in \mathbb{R})$.

Then $\forall \sigma_0 \in (0,1]$, $\quad$ ($\sigma_0 = 0$ is not needed)

and $\forall \varepsilon > 0$, $\exists$ constant $C_\varepsilon$ (depending on $\varepsilon>0$ only)

(i) $|\zeta(s)| \leq C_\varepsilon |t|^{1-\sigma_0+\varepsilon}$ for $\sigma_0 \leq \sigma$ & $|t| \geq 1$.

(ii) $|\zeta'(s)| \leq C_\varepsilon |t|^{\varepsilon}$ for $1 \leq \sigma$ & $|t| \geq 1$

Remark: In particular, one has

$$\left\{ \begin{array}{l} \zeta(1+it) = O(|t|^\varepsilon) \\ \zeta'(1+it) = O(|t|^\varepsilon) \end{array} \right. \quad \text{as } |t| \to \infty.$$

Pf of Prop2.7 :

Prop2.5 $\Rightarrow$ $$|\delta_n(s)| \leq \frac{|s|}{n^{\sigma+1}} \leq \frac{|s|}{n^{\sigma_0+1}} \quad \text{for } \sigma_0 \leq \sigma$$

And $$\delta_n(s) = \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s}\right) dx$$

also $\Rightarrow$ $$|\delta_n(s)| \leq \left|\frac{1}{n^s}\right| + \left|\frac{1}{x^s}\right| \quad \text{for } x \in [n, n+1]$$
$$\leq \frac{2}{n^\sigma} \leq \frac{2}{n^{\sigma_0}}$$

Then $\forall\ 0 \leq \delta \leq 1$
$$|\delta_n(s)| = |\delta_n(s)|^\delta |\delta_n(s)|^{1-\delta} \leq \left(\frac{|s|}{n^{\sigma_0+1}}\right)^\delta \left(\frac{2}{n^{\sigma_0}}\right)^{1-\delta}$$
$$\leq \frac{2|s|^\delta}{n^{\sigma_0+\delta}}$$

If $0 < \varepsilon \leq \sigma_0$, then $\delta = 1 - \sigma_0 + \varepsilon \leq 1$

$$\therefore \quad |\delta_n(s)| \leq \frac{2|s|^{1-\sigma_0+\varepsilon}}{n^{1+\varepsilon}}.$$

Cor2.6
$\Rightarrow$ $$|\zeta(s)| \leq \frac{1}{|s-1|} + 2|s|^{1-\sigma_0+\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \quad \forall\ \sigma \geq \sigma_0$$

For $\sigma \geq 2$, $$|\zeta(s)| \leq \sum_{n=1}^{\infty} \frac{1}{n^\sigma} \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$$

For $0 < \sigma_0 \leq \sigma < 2$ and $|t| \geq 1$, $|s| = |t| \left|\frac{\sigma}{t} + i\right| \leq 3|t|$,

$$|\zeta(s)| \leq C + \left(2 \cdot 3^{1+\varepsilon} \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}}\right) |t|^{1-\sigma_0+\varepsilon} \leq C_\varepsilon |t|^{1-\sigma_0+\varepsilon}$$

Together $\Rightarrow$ for $0<\sigma_0\le\sigma$ & $|t|\ge 1$, we have

$$|\zeta(s)|\le C_\varepsilon |t|^{1-\sigma_0+\varepsilon} \quad (\text{a new } C_\varepsilon) \quad \left(\begin{array}{c}\text{proved (i)}\\ \text{for } 0<\varepsilon\le\sigma_0\end{array}\right)$$

In particular, choosing $\varepsilon'>0$ sufficiently small, $\exists\ \delta'\le 1$ s.t.

$$|\zeta(s)|\le C_{\varepsilon'}|t|^{\delta'} \qquad \text{for } \ 0<\sigma_0\le\sigma \ \& \ |t|\ge 1$$

Hence for $\varepsilon>\sigma_0$, $\delta=1-\sigma_0+\varepsilon>1\ge\delta'$, and we have

$$|\zeta(s)|\le C_{\varepsilon'}|t|^{\delta}, \ \forall\, 0<\sigma_0\le\sigma \ \& \ |t|\ge 1$$

Therefore, we have proved that

$$\forall\,\varepsilon>0, \qquad |\zeta(s)|\le C_\varepsilon |t|^{1-\sigma_0+\varepsilon} \quad \text{on } \ 0<\sigma_0\le\sigma \ \& \ |t|\ge 1$$

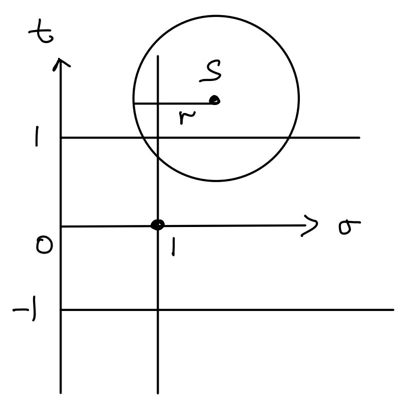
This proves (i).

To prove (ii), for $\sigma\ge 1$ & $|t|\ge 1$,

the circle $\quad s+re^{i\theta}, \ \theta\in[0,2\pi]$

with radius $r<1$ lies in the half plane

$$\{\sigma+it:\ \sigma>1-r\}$$




Take $\sigma_0=1-r$ & $\varepsilon=r$ in (i), we have

$$|\zeta(s+re^{i\theta})|\le C_r\,|t+r\sin\theta|^{1-(1-r)+r}$$

$$\le C_r'\,|t|^{2r} \qquad \text{for } |t|-r\ge 1.$$

If $|t|-r \leq 1$ then $|t| \leq 2$,

$\Rightarrow |\zeta(s+re^{i\theta})|$ is bounded (depending on $r$)

as $|s+re^{i\theta}-1| \geq |s-1|-r \geq 1-r$

Hence $|\zeta(s+re^{i\theta})| \leq C''_r |t|^{2r}$ $\forall |t| \geq 1$ ($\&$ $\sigma \geq 1$)

Then Cauchy integral formula

$$\Rightarrow \quad |\zeta'(s)| \leq \frac{1}{2\pi r} \int_0^{2\pi} |\zeta(s+re^{i\theta})| d\theta$$

$$\leq \frac{1}{r} C''_r |t|^{2r}, \quad \forall |t| \geq 1 \ \& \ \sigma \geq 1.$$

Since $1 > r > 0$ is arbitrary, we have that $\forall\, 0 < \varepsilon < 2$

$$|\zeta'(s)| \leq C_\varepsilon |t|^{\varepsilon}, \quad \forall |t| \geq 1 \ \& \ \sigma \geq 1$$

Using $|t| \geq 1$, we have $\forall \varepsilon \geq 2$,

$$|\zeta'(s)| \leq \underset{\text{(using } r=\frac{1}{2}\text{)}}{C_1 |t|} \leq C_1 |t|^{\varepsilon} \quad \forall |t| \geq 1 \ \& \ \sigma \geq 1$$

Altogether, $\forall \varepsilon > 0$, $\exists\, C_\varepsilon > 0$ s.t.

$$|\zeta'(s)| \leq C_\varepsilon |t|^{\varepsilon}, \quad \forall |t| \geq 1 \ \& \ \sigma \geq 1$$

This proves (ii).

# Ch 7 The Zeta Function and Prime Number Theorem

**Def**: The function $\pi(x)$ for $x>0$ is defined by

$$\pi(x) = \text{number of primes } p \leq x.$$

**Prime Number Theorem**

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \to \infty$$

Recall: Asymptotic relation $f(x) \sim g(x)$ as $x \to \infty$ means

that $\frac{f(x)}{g(x)} \to 1$ as $x \to \infty$.

Goal of this Chapter: use $\zeta(s)$ to prove Prime Number Theorem.

## 1. Zeros of the Zeta Function

**Relationship of $\zeta(s)$ to prime numbers:**

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}}, \qquad \text{Re}(s) > 1$$

where the infinite product is over all primes.

Pf: Fundamental theorem of Arithmetic $\Rightarrow$

$\forall n \in \{2, 3, \cdots\}$, $n = p_1^{k_1} \cdots p_m^{k_m}$ in a unique way

where $p_i$ are primes & $k_i \geq 0$ are integers.

$\Rightarrow$ For integers $M > N$,

$$\prod_{p \leq N} \left[ 1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} \cdots + \frac{1}{(p^M)^s} \right] = \sum_{p_i \leq N} \frac{1}{(p_1^{k_1} \cdots p_m^{k_m})^s}$$

with $k_i \leq M$ and $m \leq \pi(N)$

Note that $p_i \geq 2 \Rightarrow \forall\, n \leq N$,

$$n = p_1^{k_1} \cdots p_m^{k_m} \text{ for some } p_i \leq N \text{ and } k_i \leq M$$

$$\therefore \prod_{p \leq N} \left[ 1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} \cdots + \frac{1}{(p^M)^s} \right] \geq \sum_{n=1}^{N} \frac{1}{n^s}$$

On the other hand, $\displaystyle\sum_{p_i \leq N} \frac{1}{(p_1^{k_1} \cdots p_m^{k_m})^s}$ has only finitely many terms, we also have

$$\prod_{p \leq N} \left[ 1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} \cdots + \frac{1}{(p^M)^s} \right] \leq \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) .$$

Note that we have used the uniqueness of prime factorization.

Now by $\displaystyle 1 + \frac{1}{p^s} + \frac{1}{(p^2)^s} \cdots + \frac{1}{(p^M)^s} = \frac{1 - (p^{-s})^{M+1}}{1 - p^{-s}}$

we have
$$\sum_{n=1}^{N} \frac{1}{n^s} \leq \prod_{p \leq N} \frac{1 - (p^{-s})^{M+1}}{1 - p^{-s}} \leq \zeta(s)$$

Letting $M \to \infty$ $(M > N) \Rightarrow \displaystyle\sum_{n=1}^{N} \frac{1}{n^s} \leq \prod_{p \leq N} \frac{1}{1 - p^{-s}} \leq \zeta(s)$.

Letting $N \to \infty$, we proved the Relation for $s > 1$. Then uniqueness of analytic continuation implies it holds for $\mathrm{Re}\, s > 1$. #

$\boxed{\text{Thm 1.1 The only zeros of } \zeta(s) \text{ outside } 0 \le \mathrm{Re}(s) \le 1, \text{ the critical strip, are } -2, -4, -6, \cdots .}$

Pf: For $\mathrm{Re}(s) > 1$, $\zeta(s) = \prod_p \frac{1}{1-p^{-s}} \neq 0$.

For $\mathrm{Re}(s) < 0$, we use the functional equation

$$\xi(s) = \xi(1-s),$$

where

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\tfrac{s}{2}\right) \zeta(s).$$

Rewrite the functional equation as

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \cdot \pi^{-\frac{(1-s)}{2}} \Gamma\left(\tfrac{1-s}{2}\right) \zeta(1-s).$$

- $\mathrm{Re}(s) < 0 \Rightarrow \mathrm{Re}(1-s) > 1 \Rightarrow \zeta(1-s) \neq 0$
- clearly $\Gamma\left(\frac{1-s}{2}\right) \neq 0$ & $\pi^{s-\frac{1}{2}} \neq 0$
- and by Thm 1.6 $1/\Gamma\left(\frac{s}{2}\right)$ has zeros at $\frac{s}{2} = 0, -1, -2, \cdots$

All together, the zeros of $\zeta(s)$ in $\mathrm{Re}(s) < 0$ are exactly $s = -2, -4, -6, \cdots$. ※

Remarks:

(i) $\boxed{\text{Riemann hypothesis: The zeros of } \zeta(s) \text{ in the critical strip lie on the line } \mathrm{Re}(s) = \tfrac{1}{2}.}$

(ii) $s = -2, -4, -6, \cdots$ are called the trivial zeros of $\zeta(s)$.

**Thm 1.2** $\zeta(1+it) \neq 0, \quad \forall t$

Remark: the pole $s=1$ (i.e. $t=0$) is included.

The proof needs some lemmas.

**Lemma 1.3** If $\mathrm{Re}(s)>1$, then

$$\log \zeta(s) = \sum_{p,m} \frac{1}{m} p^{-sm} = \sum_{n=1}^{\infty} \frac{c_n}{n^s}$$

for some $c_n \geq 0$.

Pf: For $s>1$,

$$\log \zeta(s) = \log \prod_p \frac{1}{1-p^{-s}} = \sum_p \log \frac{1}{1-p^{-s}}$$

$$= \sum_p \sum_{m=1}^{\infty} \frac{1}{m} (p^{-s})^m \qquad (\text{since } p^{-s} < p^{-1} < 1\,)$$

Since the double sum converges absolutely, we have

$$\log \zeta(s) = \sum_{p,m} \frac{1}{m} p^{-sm} .$$

Clearly, the absolute convergence of the double sum holds for

$$\mathrm{Re}(s) > 1 \qquad (\,|p^{-s}| = p^{-\mathrm{Re}\, s} < p^{-1} < 1\,),$$

the RHS defines a holo. function on $\mathrm{Re}(s)>1$.

Then uniqueness of analytic continuation $\Longrightarrow$

$$\log \zeta(s) = \sum_{p,m} \frac{1}{m} p^{-sm} \quad \forall\, \mathrm{Re}(s)>1 .$$

Note that the general term of the sum is $\frac{1}{m}(p^m)^{-s}$, we have $\log \zeta(s) = \sum_{n=1}^{\infty} c_n n^{-s}$ with

$$c_n = \begin{cases} \frac{1}{m} &, \text{ if } n = p^m \text{ for some prime } p \\ 0 &, \text{ otherwise.} \end{cases}$$

※

**Lemma 1.4** $\forall \theta \in \mathbb{R}$, $3 + 4\cos\theta + \cos 2\theta \geq 0$

Pf: $3 + 4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2$. ※

**Cor 1.5** If $s = \sigma + it$ with $\sigma > 1$ & $t \in \mathbb{R}$,

then $\log |\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \geq 0$

Pf: $\log |\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)|$

$= 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2it)|$

$= 3\operatorname{Re}[\log\zeta(\sigma)] + 4\operatorname{Re}[\log\zeta(\sigma+it)] + \operatorname{Re}[\log\zeta(\sigma+2it)]$

By Lemma 1.3

$= 3\sum_n c_n \operatorname{Re}(n^{-\sigma}) + 4\sum_n c_n \operatorname{Re}(n^{-(\sigma+it)}) + \sum_n c_n \operatorname{Re}(n^{-(\sigma+2it)})$

$= \sum_n c_n \left(3n^{-\sigma} + 4\operatorname{Re} e^{-(\sigma+it)\log n} + \operatorname{Re} e^{-(\sigma+2it)\log n}\right)$

$= \sum_n c_n [3n^{-\sigma} + 4n^{-\sigma}\cos(t\log n) + n^{-\sigma}\cos(2t\log n)]$

$= \sum_n c_n n^{-\sigma}[3 + 4\cos(t\log n) + \cos(2t\log n)]$

$\geq 0$ by Lemma 1.4 (& Lemma 1.3 that $c_n \geq 0$) ※

Pf of Thm 1.2

Suppose on the contrary that

$$\zeta(1+it_0)=0 \quad \text{for some } t_0\neq 0.$$

We consider the 3 factors in Cor 1.5 for $\sigma\to 1$ & $t=t_0$.

Since $\zeta(s)$ is holo. near $s=1+it_0$, $t_0\neq 0$,

$$\zeta(s)=(s-(1+it_0))^m h(s) \quad \text{near } s=1+it_0$$

with • $m\geq 1$

• $h(s)$ holo near $s=1+it_0$ and $h(1+it_0)\neq 0$.

Hence for some const. $C>0$,

$$(*)_1 \qquad |\zeta(\sigma+it_0)|^4 \leq C(\sigma-1)^4 \quad \text{as } \sigma\to 1. \qquad (\sigma>1)$$

Then using $s=1$ is a simple pole of $\zeta(s)$, we also have

$$(*)_2 \qquad |\zeta(\sigma)|^3 \leq \frac{C_1}{(\sigma-1)^3} \quad \text{as } \sigma\to 1 \quad (\sigma>1)$$

Finally, $\zeta(s)$ holo. near $s=1+2it_0$,

$$(*)_3 \qquad |\zeta(\sigma+2it_0)| \leq C_2 \quad \text{as } \sigma\to 1 \quad (\sigma>1)$$

Combining $(*)_1$, $(*)_2$, $(*)_3$ and Cor 1.5, we have

$$1\leq |\zeta^3(\sigma)\zeta^4(\sigma+it_0)\zeta(\sigma+2it_0)| \longrightarrow 0 \quad \text{as } \sigma\to 1 \quad (\sigma>1)$$

which is a contradiction. The proof is completed. ※

## 1.1 Estimates for $1/\zeta(s)$

**Prop 1.6** $\forall \varepsilon > 0$, $\exists\, C_\varepsilon > 0$ s.t.

$$\frac{1}{|\zeta(s)|} \leq C_\varepsilon |t|^\varepsilon \quad \text{for } s = \sigma + it,\ \sigma \geq 1 \text{ and } |t| \geq 1.$$

Pf: By Cor 1.5 and $\zeta(s)$ only has a pole at $s=1$, we have

$$|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \geq 1, \quad \forall\ \sigma \geq 1$$

By Prop 2.7(i) of Ch6, (taking $\sigma_0 = 1$)

$$|\zeta(\sigma+2it)| \leq C_1 |t|^\varepsilon \qquad \forall\ \sigma \geq 1\ \&\ |t| \geq 1. \quad (C_1 = C_1(\varepsilon) > 0)$$

Hence

$$1 \leq |\zeta^3(\sigma)\zeta^4(\sigma+it)| \cdot C_1 |t|^\varepsilon$$

Then similar to $(*)_2$ in the proof of Thm 1.2,

$$|\zeta^3(\sigma)| \leq \frac{C_2}{(\sigma-1)^3} \quad \text{for } \sigma > 1. \qquad (C_3 = C_3(\varepsilon) > 0)$$

Hence $$|\zeta^4(\sigma+it)| \geq \frac{C_3(\sigma-1)^3}{|t|^\varepsilon} \quad \forall\ \sigma > 1\ \&\ |t| \geq 1$$

and clearly this inequality trivially holds for $\sigma = 1$.

Hence

$$(3) \quad |\zeta(\sigma+it)| \geq C_4 (\sigma-1)^{\frac{3}{4}} |t|^{-\frac{\varepsilon}{4}}, \quad \forall\ \sigma \geq 1\ \&\ |t| \geq 1$$

$$(C_4 = C_4(\varepsilon) > 0)$$

Note that by Prop 2.7 (ii) of Ch 6, we have

for $\sigma' > \sigma \geq 1$,

$$|\zeta(\sigma'+it) - \zeta(\sigma+it)| \leq |\zeta'(\sigma_c+it)||\sigma'-\sigma| \quad \text{for some } \sigma \leq \sigma_c \leq \sigma'$$

$$\leq C_5 |t|^{\varepsilon} |\sigma'-\sigma| \quad (C_5 = C_5(\varepsilon) > 0)$$

$$\leq C_5 |t|^{\varepsilon} (\sigma'-1). \quad (\sigma' > \sigma \geq 1)$$

Let $A = \left(\frac{C_4}{2C_5}\right)^4 > 0$.

<u>Case 1</u> $\sigma - 1 \geq A|t|^{-5\varepsilon}$

Then (3) $\Rightarrow$ $|\zeta(\sigma+it)| \geq C_4 (A|t|^{-5\varepsilon})^{\frac{3}{4}} |t|^{-\frac{\varepsilon}{4}}$

$$= (C_4 A^{\frac{3}{4}}) |t|^{-4\varepsilon}$$

<u>Case 2</u> $\sigma - 1 < A|t|^{-5\varepsilon}$

Take $\sigma' > \sigma$ such that $\sigma' - 1 = A|t|^{-5\varepsilon}$.

Then triangle inequality $\Rightarrow$

$$|\zeta(\sigma+it)| \geq |\zeta(\sigma'+it)| - |\zeta(\sigma'+it) - \zeta(\sigma+it)|$$

$$\geq C_4 (\sigma'-1)^{\frac{3}{4}} |t|^{-\frac{\varepsilon}{4}} - C_5 |t|^{\varepsilon} (\sigma'-1)$$

$$= \left[C_4 (\sigma'-1)^{-\frac{1}{4}} |t|^{-\frac{\varepsilon}{4}} - C_5 |t|^{\varepsilon}\right] (\sigma'-1)$$

$$= \left[C_4 \frac{1}{(A|t|^{-5\varepsilon})^{\frac{1}{4}}} |t|^{-\frac{\varepsilon}{4}} - C_5 |t|^{\varepsilon}\right] (\sigma'-1)$$

$$= \left[ C_4 \cdot \frac{2C_5}{C_4} \cdot |t|^{\varepsilon} - C_5 |t|^{\varepsilon} \right] (\sigma' - 1)$$

$$= C_5 |t|^{\varepsilon} (\sigma' - 1)$$

$$= C_5 A \, |t|^{-4\varepsilon}$$

Hence $\forall \varepsilon > 0$, $\quad |\zeta(\sigma + it)| \geq C_\varepsilon |t|^{-4\varepsilon}$

where $\quad C_\varepsilon = \min\{ C_4 A^{\frac{3}{4}}, C_5 A \}$.

Replacing $4\varepsilon$ by $\varepsilon$, we have

$|\zeta(\sigma + it)| \geq C_\varepsilon |t|^{-\varepsilon}$ with a new $C_\varepsilon$. #