§2 The Zeta Function

Def: The Riemann Zeta Function for $s>1$ ($s\in\mathbb{R}$) is defined by
$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

2.1 Functional Equation & Analytic Continuation

Prop 2.1 $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ converges for $Re(s)>1$, and holomaphic in $\{s: Re(s)>1\}$

Pf: Let $s=\sigma+it$ $(\sigma,t\in\mathbb{R})$.

Then $$\left|\frac{1}{n^s}\right| = e^{Re(-s\log n)} = e^{-\sigma\log n} = \frac{1}{n^\sigma}$$

$\Rightarrow$ $\forall\,\delta>0$, then for $\sigma>1+\delta$,

$$\left|\frac{1}{n^s}\right| \leq \frac{1}{n^{1+\delta}} \text{ and } \sum_{n=1}^{\infty}\frac{1}{n^{1+\delta}}<+\infty \Rightarrow$$

the series $\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$ converges absolutely uniformly and hence define a holo. function on $\{s=\sigma+it : \sigma>1+\delta\}$.

Since $\delta>0$ is arbitrary, $\zeta(s)$ is defined and holomaphic on $\{s=\sigma+it : \sigma>1\}$. ※

Recall: The Theta Function defined for $t > 0$ by

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$$

satisfies $\boxed{\vartheta(t) = t^{-\frac{1}{2}} \vartheta\left(\frac{1}{t}\right)}$ (Application (1) of Poisson summation formula) (Thm 2.4 in Ch 4)

Note that $\vartheta(t) - 1 = 2 \sum_{n=1}^{\infty} e^{-\pi n^2 t}$

$$\Rightarrow \quad |\vartheta(t) - 1| \le 2 \sum_{n=1}^{\infty} \left(e^{-\pi t}\right)^n = \frac{2e^{-\pi t}}{1 - e^{-\pi t}} \quad (\text{for } t > 0)$$

Hence $\exists\, C > 0$ s.t.

$$|\vartheta(t) - 1| \le C e^{-\pi t} \quad \text{for } t \ge 1.$$

Then
$$\vartheta(t) \le t^{-\frac{1}{2}} \left(1 + C e^{-\frac{\pi}{t}}\right) \quad \text{for } t < 1$$
$$\le C t^{-\frac{1}{2}} \quad \text{as } t \to 0.$$

In summary

$$\boxed{\begin{cases} \vartheta(t) \le C t^{-\frac{1}{2}} & \text{as } t \to 0 \\ |\vartheta(t) - 1| \le C e^{-\pi t} & \text{as } t \to \infty \end{cases}}$$

Thm 2.2 If $\mathrm{Re}(s) > 1$, then

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \frac{1}{2} \int_0^{\infty} u^{\frac{s}{2} - 1} (\vartheta(u) - 1)\, du$$

Pf : By properties summarized above, for $\operatorname{Re} s > 1$

$$\left| u^{\frac{s}{2}-1}(\vartheta(u)-1) \right| \leq u^{\frac{\operatorname{Re}s}{2}-1} |\vartheta(u)-1|$$

$$\leq \begin{cases} C\, u^{\frac{\operatorname{Re}s-1}{2}-1} & \text{as } u \to 0 \\ C\, u^{\frac{\operatorname{Re}s}{2}-1} e^{-\pi u} & \text{as } u \to \infty \end{cases}$$

$\therefore \frac{1}{2}\int_0^\infty u^{\frac{s}{2}-1}(\vartheta(u)-1)\,du$ converges absolutely

and hence

$$= \int_0^\infty u^{\frac{s}{2}-1}\left(\frac{\vartheta(u)-1}{2}\right) du$$

$$= \int_0^\infty u^{\frac{s}{2}-1}\left(\sum_{n=1}^{\infty} e^{-\pi n^2 u}\right) du$$

$$= \sum_{n=1}^{\infty} \int_0^\infty u^{\frac{s}{2}-1} e^{-\pi n^2 u}\, du$$

(change of variable $u = \frac{t}{\pi n^2}$)

$$= \sum_{n=1}^{\infty} \int_0^\infty \left(\frac{t}{\pi n^2}\right)^{\frac{s}{2}-1} e^{-t} \cdot \frac{dt}{\pi n^2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{(\pi n^2)^{s/2}} \int_0^\infty e^{-t} t^{\frac{s}{2}-1}\, dt$$

$$= \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \sum_{n=1}^{\infty} \frac{1}{n^s}$$

$$= \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

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Def The __Xi Function__ $\xi(s)$ is defined by

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma(\tfrac{s}{2}) \zeta(s)$$

Thm 2.3

- $\xi(s)$ is holo. for $\mathrm{Re}(s) > 1$
- $\xi(s)$ has analytic continuation as a meromorphic function to $\mathbb{C}$ with simple poles at $s=0$ & $s=1$.

  (with $\mathrm{res}_{s=0}\, \xi(s) = -1$, $\mathrm{res}_{s=1}\, \xi(s) = 1$)
- And

$$\xi(s) = \xi(1-s), \quad \forall s \in \mathbb{C} \setminus \{0,1\}$$

Pf: To simplify notation, let $\psi(u) = \frac{1}{2}(\vartheta(u) - 1)$.

Then $\quad \vartheta(u) = 1 + 2\psi(u)$

$$\Rightarrow \quad 1 + 2\psi(u) = \vartheta(u) = u^{-\frac{1}{2}} \vartheta(\tfrac{1}{u})$$

$$= u^{-\frac{1}{2}}(1 + 2\psi(\tfrac{1}{u}))$$

$$\therefore \quad \psi(u) = u^{-\frac{1}{2}} \psi(\tfrac{1}{u}) + \frac{1}{2u^{\frac{1}{2}}} - \frac{1}{2}, \quad \forall u \in (0,\infty)$$

By Thm 2.2, for $\mathrm{Re}\, s > 1$, we have

$$\xi(s) = \int_0^\infty u^{\frac{s}{2}-1} \psi(u)\, du$$

$$= \int_0^1 u^{\frac{s}{2}-1}\left[u^{-\frac{1}{2}}\psi(\tfrac{1}{u}) + \frac{1}{2u^{\frac{1}{2}}} - \frac{1}{2}\right] du + \int_1^\infty u^{\frac{s}{2}-1} \psi(u)\, du$$

$$= \frac{1}{2}\int_0^1 u^{\frac{s-1}{2}-1}\,du - \frac{1}{2}\int_0^1 u^{\frac{s}{2}-1}\,du$$

$$+ \int_0^1 u^{\frac{s}{2}-\frac{3}{2}}\psi\left(\frac{1}{u}\right)du + \int_1^\infty u^{\frac{s}{2}-1}\psi(u)\,du$$

$$= \frac{1}{2}\left[\frac{u^{\frac{s-1}{2}}}{\frac{s-1}{2}}\right]_0^1 - \frac{1}{2}\left[\frac{u^{\frac{s}{2}}}{\frac{s}{2}}\right]_0^1 + \int_\infty^1 u^{-\frac{s}{2}+\frac{3}{2}}\psi(u)\left(-\frac{du}{u^2}\right)$$

$$+ \int_1^\infty u^{\frac{s}{2}-1}\psi(u)\,du$$

$$\therefore \quad \xi(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \left(u^{-\frac{s}{2}-\frac{1}{2}} + u^{\frac{s}{2}-1}\right)\psi(u)\,du \quad \text{---} (*)$$

Note that $|\psi(u)| = \frac{1}{2}|\vartheta(u)-1| \le Ce^{-\pi u}$ as $u \to +\infty$

$\therefore \int_1^\infty (u^{-\frac{s}{2}-\frac{1}{2}} + u^{\frac{s}{2}-1})\psi(u)\,du$ converges absolutely

for all $s \in \mathbb{C}$ (not just $\mathrm{Re}\, s > 1$) and defines an entire function. Hence the RHS of $(*)$ is a meromorphic function with simple poles at $s=0$ & $s=1$ (with corresponding residues). This proves the first two statements.

For the last formula, substitute $1-s$ in $(*)$, we have

$$\xi(1-s) = \frac{1}{(1-s)-1} - \frac{1}{(1-s)} + \int_1^\infty \left(u^{-\frac{1-s}{2}-\frac{1}{2}} + u^{\frac{1-s}{2}-1}\right)\psi(u)\,du$$

$$= -\frac{1}{s} + \frac{1}{s-1} + \int_1^\infty \left(u^{\frac{s}{2}-1} + u^{-\frac{s}{2}-\frac{1}{2}}\right)\psi(u)\,du$$

$$= \xi(s).$$

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