§4 Weierstrass Infinite Products

Thm 4.1 Given any seq $\{a_n\} \subset \mathbb{C}$ with $|a_n| \to +\infty$ as $n \to +\infty$,

$\exists$ entire function f such that

$$\begin{cases} f(a_n) = 0, & \forall n, \\ f(z) \neq 0, & \forall z \in \mathbb{C} \setminus \{a_n\} \end{cases}$$

If g is another entire function with the same property, then $\exists$ entire function $h(z)$ such that

$$g(z) = f(z)\, e^{h(z)}.$$

Pf: The 2nd statement is easy to prove: near $z = a_n$

$$\frac{g(z)}{f(z)} = \frac{(z-a_n)^m g_1(z)}{(z-a_n)^m f_1(z)}$$ where f_1, g_1 holo. near a_n and $f_1(a_n) \neq 0$, $g_1(a_n) \neq 0$

$$= \frac{g_1(z)}{f_1(z)}$$ holo. near a_n

$\therefore$ $\{a_n\}$ are removable singularities of $\frac{g(z)}{f(z)}$

Since f & g have no other zeros, $g(z)/f(z)$ is entire with no zero. Therefore $\frac{g(z)}{f(z)} = e^{h(z)}$

for some entire function $h(z)$. ($\mathbb{C}$ is simply-connected) #

To prove the 1^{st} statement, we need a lemma concerning <u>canonical factors</u>:

$$\begin{cases} E_0(z) = 1 - z \quad \& \\ E_k(z) = (1-z)\, e^{z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}}, \quad k \geq 1 \end{cases}$$

(k = degree of the canonical factor)

<u>Lemma 4.2</u> $\exists\, C > 0$ such that $\forall\, k \geq 0$, $\lvert 1 - E_k(z) \rvert \leq C \lvert z \rvert^{k+1}$ for $z \in \overline{D}_{\frac{1}{2}}(0)$

<u>Pf</u>: On $\overline{D}_{\frac{1}{2}}(0)$, $\log(1-z)$ well-defined

$$\Rightarrow E_k(z) = (1-z)\, e^{z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}} \qquad (k \geq 1)$$

$$= e^{\log(1-z) + z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}}$$

By Taylor's expansion of $\log(1-z)$,

$$w \overset{\text{denote}}{=} \log(1-z) + z + \frac{z^2}{2} + \cdots + \frac{z^k}{k} = -\sum_{n=k+1}^{\infty} \frac{z^n}{n} = -z^{k+1} \sum_{n=k+1}^{\infty} \frac{z^{n-k-1}}{n}$$

$$= -z^{k+1} \sum_{j=0}^{\infty} \frac{z^j}{k+1+j} \qquad (j = n-k-1)$$

$$\Rightarrow \lvert w \rvert \leq \lvert z \rvert^{k+1} \sum_{j=0}^{\infty} \lvert z \rvert^j \leq \lvert z \rvert^{k+1} \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j = 2 \lvert z \rvert^{k+1} \left(\leq \frac{1}{2^k}\right)$$

$\therefore\ |1-E_k(z)| = |1-e^w| \le c'|w|$ for some $c'>0$ (indep. of k)

$\le 2c'|z|^{k+1}$. (Then one easily includes the case: $k=0$) $\#$

<u>Pf of the 1st statement of Thm 4.1</u>

"If" 0 is a "m-order zero" of f ← to be found (m could be 0, ie $f(0)\ne 0$)

we remove those $a_{n_1} = \cdots = a_{n_m} = 0$ from the seq $\{a_n\}$.

For simplicity, denote the subseq. by $\{a_n\}$ again.

Then consider the infinite product.

$$f(z) = z^m \prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right).$$

For any fixed $R>0$, by re-arranging finitely many terms,

we may assume $\begin{cases} |a_n| \le 2R & \text{for } n=1,\cdots,n_0-1 \text{ and} \\ |a_n| > 2R & \text{for } n \ge n_0 \end{cases}$

(as $|a_n| \to +\infty$)

$\forall z \in D_R$, we have $\left|\frac{z}{a_n}\right| < \frac{1}{2}$ for $n \ge n_0$

By Lemma 4.2, $\left|1 - E_n\left(\frac{z}{a_n}\right)\right| \le c\left|\frac{z}{a_n}\right|^{n+1}$ for some $c>0$ indep. of n

$\le \frac{c}{2^{n+1}}$

$\Rightarrow \sum_{n=n_0}^{\infty} \left|1 - E_n\left(\frac{z}{a_n}\right)\right|$ is convergent.

Hence Prop 3.2 $\Rightarrow$

$$\prod_{n=n_0}^{\infty} E_n\left(\frac{z}{a_n}\right) = \prod_{n=n_0}^{\infty}\left[1+\left(E_n\left(\frac{z}{a_n}\right)-1\right)\right]$$ converges uniformly on D_R

$\Rightarrow$ $\prod_{n=n_0}^{\infty} E_n\left(\frac{z}{a_n}\right)$ is a holo. function on D_R

and Prop 3.1 $\Rightarrow$ $\prod_{n=n_0}^{\infty} E_n\left(\frac{z}{a_n}\right) \neq 0 \quad \forall z \in D_R$

$$\therefore \quad f(z) = z^m \prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right) = z^m \prod_{n=1}^{n_0-1} E_n\left(\frac{z}{a_n}\right) \cdot \prod_{n=n_0}^{\infty} E_n\left(\frac{z}{a_n}\right)$$

is holo on D_R with only those zeros

at $z=0$ (if $m>0$) or $z=a_n$ with $|a_n|<R$.

As $R>0$ is arbitrary, $f(z)$ converges locally uniformly to an entire function with exactly the zeros prescribed by the sequence $\{a_n\}$ (and m-order zero at 0)

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§5 Hadamard's Factorization Theorem

Thm 5.1 Suppose f entire, ρ_f = order of growth of f.

Let $k \in \mathbb{N}$ such that $k \leq \rho_f < k+1$.

If $a_1, a_2 \cdots$ are the non-zero zeros of f, then

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)$$

where P is a polynomial of $\deg P \leq k$, and

m = order of zero of f at $z=0$ (could have $m=0$)

Remark: Recall that $E_k(z) = (1-z)e^{z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}}$.

The different in Weierstrass & Hadamard is that k is fixed in Hadamard, independent of n, in $E_k\left(\frac{z}{a_n}\right)$, degree of the poly in the exponential in the facta $= k$.

But in Weierstrass, it is $E_n\left(\frac{z}{a_n}\right)$, the poly in the exponential in the facta has degree $\to +\infty$.

To prove Hadamard's Theorem, we start with some lemmas.

Conditions and notations as in the Thm.

Lemma 5.2:
$$|E_k(z)| \geq \begin{cases} e^{-C|z|^{k+1}} & \text{if } |z| \leq \frac{1}{2} \\ |1-z|\, e^{-C'|z|^k} & \text{if } |z| \geq \frac{1}{2} \end{cases}$$

for some constants C & $C' > 0$. (C' depends on k)

Pf: If $|z| \leq \frac{1}{2}$, $\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}$ holds and hence

$$E_k(z) = (1-z)\, e^{z + \frac{z}{2} + \cdots + \frac{z^k}{k}}$$

$$= e^{-\sum_{n=k+1}^{\infty} \frac{z^n}{n}}$$

Let $w = -\sum_{n=k+1}^{\infty} \frac{z^n}{n}$ again, we have as in the proof of Weierstrass' Thm, we have

$$|w| \leq C|z|^{k+1} \quad \text{for some } C > 0.$$

Hence $|E_k(z)| = |e^w| = e^{\mathrm{Re}\, w} \geq e^{-|w|} \geq e^{-C|z|^{k+1}}$

If $|z| > \frac{1}{2}$, then

$$|E_k(z)| = |1-z|\left|e^{z + \cdots + \frac{z^k}{k}}\right|$$

$$\geq |1-z|\, e^{-\left|z + \cdots + \frac{z^k}{k}\right|}$$

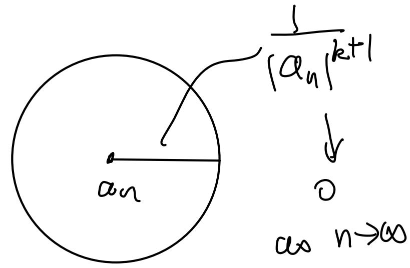
$$\geq |1-z|\, e^{-C'|z|^k}.$$

for some C' (depending on k) since $|z| > \frac{1}{2}$ #

Lemma 5.3 $\forall\, s$ s.t. $\rho_f < s < k+1$, $\exists$ const. $C>0$ s.t.

$$\left|\prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)\right| \geq e^{-C|z|^s}$$

for $z \in \mathbb{C} \setminus \bigcup_{n=1}^{\infty} B_{\frac{1}{|a_n|^{k+1}}}(a_n)$.



(In the following, C means a constant indep. of z, may be different in each step.)

Pf

Step 1: For any $z \in \mathbb{C}$,

$$\left|\prod_{|a_n|>2|z|} E_k\left(\frac{z}{a_n}\right)\right| \geq e^{-C|z|^s} \quad \text{for some } C>0.$$

Pf of Step 1:

$$\left|\prod_{|a_n|>2|z|} E_k\left(\frac{z}{a_n}\right)\right| = \prod_{|a_n|>2|z|} \left|E_k\left(\frac{z}{a_n}\right)\right|$$

(Convergence will be proved later in the proof of Hadamard Thm.)

$$\geq \prod_{|a_n|>2|z|} e^{-C\left|\frac{z}{a_n}\right|^{k+1}}$$

by Lemma 5.2 & $\left|\frac{z}{a_n}\right| < \frac{1}{2}$

$$= e^{-C|z|^{k+1} \sum_{|a_n|>2|z|} \frac{1}{|a_n|^{k+1}}}$$

$$s<k+1 \Rightarrow \frac{1}{|a_n|^{k+1}} = \frac{1}{|a_n|^s\,|a_n|^{k+1-s}} \leq \frac{1}{|a_n|^s} \cdot \frac{1}{2^{k+1-s}\,|z|^{k+1-s}}$$

$\rho_f < s \Rightarrow$ (by Thm 2.1 of Ch5) $\sum \frac{1}{|a_n|^s} < +\infty$.

$$\therefore \quad \sum_{|a_n|>2|z|} \frac{1}{|a_n|^{k+1}} \leq C \frac{1}{|z|^{k+1-s}}$$ (note this C is not the same C above)

Hence $$\left| \prod_{|a_n|>2|z|} E_k\left(\frac{z}{a_n}\right) \right| \geq e^{-C|z|^s}$$ (note this C is the product of the 2 different C's above)

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<u>Step 2</u> For $z \in \mathbb{C} \setminus \bigcup_{n=1}^{\infty} B_{\frac{1}{|a_n|^{k+1}}}(a_n)$ and $|z| \geq \frac{1}{2} \min |a_n|$

$$\prod_{|a_n| \leq 2|z|} \left| 1 - \frac{z}{a_n} \right| \geq e^{-C|z|^s}$$ for some $C > 0$.

<u>Pf of Step 2</u>:

For $z \in \mathbb{C} \setminus \bigcup_{n=1}^{\infty} B_{\frac{1}{|a_n|^{k+1}}}(a_n)$,

$$|z - a_n| \geq \frac{1}{|a_n|^{k+1}}$$

$$\Rightarrow \prod_{|a_n| \leq 2|z|} \left| 1 - \frac{z}{a_n} \right| \geq \prod_{|a_n| \leq 2|z|} \frac{1}{|a_n|^{k+2}}$$

$$= e^{-(k+2) \sum_{|a_n| \leq 2|z|} \log|a_n|}$$

Note that $(k+2) \sum_{|a_n| \leq 2|z|} \log|a_n| \leq (k+2) \sum_{|a_n| \leq 2|z|} \log(2|z|)$

$$\leq (k+2) \log(2|z|) \, \pi(2(1+\delta)|z|)$$ (for any $\delta > 0$)

Then for any s_1 s.t. $\rho_f < s_1 < s$, we have (by Thm 2.1)

$$n(2(1+\delta)|z|) \le C\left(2(1+\delta)|z|\right)^{s_1}$$

$$\le C|z|^{s_1} \quad (\text{for some } C>0)$$

Hence

$$(k+2)\sum_{|a_n|\le 2|z|} \log|a_n| \le (k+2)\cdot\log(2|z|)\cdot C|z|^{s_1}$$

$$\le C|z|^{s} \quad \text{since } s>s_1 \text{ and}$$

$$|z| \ge \tfrac{1}{2}\min|a_n|.$$

$$\therefore \prod_{|a_n|\le 2|z|}\left|1-\frac{z}{a_n}\right| \ge e^{-(k+2)\sum_{|a_n|\le 2|z|}\log|a_n|}$$

$$\ge e^{-C|z|^{s}}, \quad \text{for some } C>0.$$

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<u>Step 3</u>: For $z\in\mathbb{C}\setminus\bigcup_{n=1}^{\infty} B_{\frac{1}{|a_n|^{k+1}}}(a_n)$, and $|z|\ge\frac{1}{2}\min|a_n|$,

$$\left|\prod_{|a_n|\le 2|z|} E_k\left(\frac{z}{a_n}\right)\right| \ge e^{-C|z|^{s}} \quad \text{for some } C>0.$$

<u>Pf of Step 3</u>: By lemma 5.2

$$\left|\prod_{|a_n|\le 2|z|} E_k\left(\frac{z}{a_n}\right)\right| = \prod_{|a_n|\le 2|z|}\left|E_k\left(\frac{z}{a_n}\right)\right|$$

$$\ge \prod_{|a_n|\le 2|z|}\left(\left|1-\frac{z}{a_n}\right| e^{-c'\left|\frac{z}{a_n}\right|^{k}}\right)$$

$$= \left(\prod_{|a_n| \le 2|z|} \left| 1 - \frac{z}{a_n} \right| \right) \left(\prod_{|a_n| \le 2|z|} e^{-c' \left| \frac{z}{a_n} \right|^k} \right)$$

(by Step 2) $\geq e^{-c|z|^s} \, e^{-c'|z|^k \sum_{|a_n| \le 2|z|} \frac{1}{|a_n|^k}}$

Since $k \le \rho_f < s$, for $|a_n| \le 2|z|$

$$|a_n|^k = \frac{|a_n|^s}{|a_n|^{s-k}} \geq \frac{|a_n|^s}{(2|z|)^{s-k}}$$

$$\Rightarrow \sum_{|a_n| \le 2|z|} \frac{1}{|a_n|^k} \le c|z|^{s-k} \sum \frac{1}{|a_n|^s} \le C|z|^{s-k}$$ (C's are diffenet from each others)

Hence $\left| \prod_{|a_n| \le 2|z|} E_k\left(\frac{z}{a_n}\right) \right| \geq e^{-C|z|^s} \, e^{-c\,|z|^k |z|^{s-k}}$

$= e^{-C|z|^s}$ for some $C > 0$.

Step 4: Complete the proof of the lemma 5.3.

$\forall z \in \mathbb{C} \setminus B_{\frac{1}{|a_n|^{k+1}}}(a_n)$

If $|z| < \frac{1}{2} \min |a_n|$, then

$$\prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right) = \prod_{|a_n| > 2|z|} E_k\left(\frac{z}{a_n}\right).$$

Then Step 1 $\Rightarrow \left| \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right) \right| \geq e^{-C|z|^s}$ for some $c > 0$.

If $|z| \geq \frac{1}{2} \min |a_n|$, then

$$\prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right) = \prod_{|a_n| \leq 2|z|} E_k\left(\frac{z}{a_n}\right) \cdot \prod_{|a_n| > 2|z|} E_k\left(\frac{z}{a_n}\right)$$

Steps 1 & 3 $\Rightarrow$ $\left|\prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)\right| \geq e^{-c|z|^s} e^{-c|z|^s}$

$= e^{-c|z|^s}$ (all c are diff)

※

Cor 5.4 $\exists$ a sequence $\{r_m\}$ with $r_m \to +\infty$ as $m \to +\infty$

(may choose $\{r_m\}$ to be increasing)

such that

$$\left|\prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)\right| \geq e^{-c|z|^s} \quad \text{for } |z| = r_m$$

for some constant $c > 0$.

Pf Since $k+1 > \rho_f$, $\sum_n \frac{1}{|a_n|^{k+1}} < +\infty$,

$\exists\, N > 0$ s.t. $\sum_{n=N}^{\infty} \frac{1}{|a_n|^{k+1}} < \frac{1}{10}$.

Consider intervals $I_n = \left[|a_n| - \frac{1}{|a_n|^{k+1}},\ |a_n| + \frac{1}{|a_n|^{k+1}}\right]$.

Then $|I_n| = \frac{2}{|a_n|^{k+1}}$.

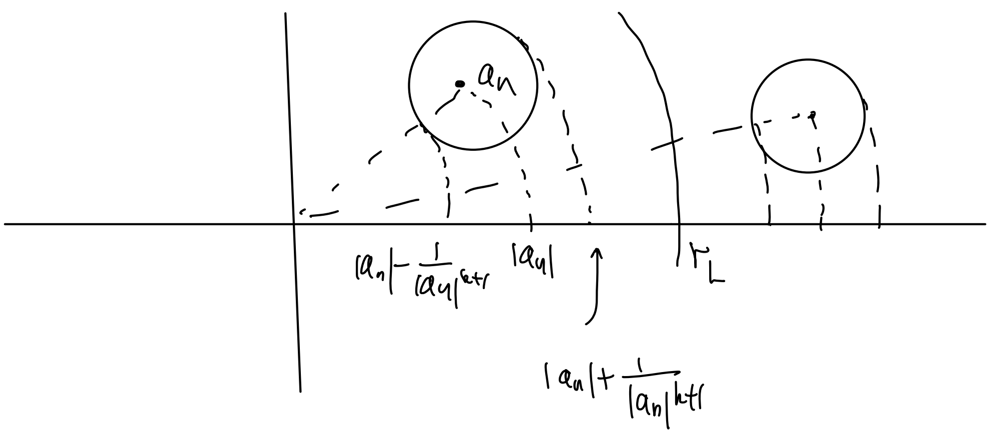
$$\Rightarrow \sum_{n=N}^{\infty} |I_n| = 2 \sum_{n=N}^{\infty} \frac{1}{|a_n|^{k+1}} < \frac{1}{5}.$$

Hence $\bigcup_{n=N}^{\infty} I_n$ cannot cover an interval of length 1.

$\Rightarrow \bigcup_{n=N}^{\infty} I_n$ cannot cover the interval $[L, L+1]$

(for sufficiently large $L \in \mathbb{N}$).

$\Rightarrow \exists\, r_L$ s.t. $|z| = r_L \Rightarrow z \notin B_{\frac{1}{|a_n|^{k+1}}}(a_n)$, $\forall n \geq 1$



By Lemma 5.3 $\quad \left| \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right) \right| \geq e^{-C|z|^s}, \quad \forall |z| = r_L,$

($L \in \mathbb{N}$ sufficiently large)

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Lemma 5.5 Suppose g is entire, if $\exists$ seq. $\{r_m\}$, $r_m \to +\infty$ s.t.

$$\operatorname{Re} g(z) \leq C r_m^s \quad \text{for } |z| = r_m, \ \forall m \geq 1.$$

Then g is a polynomial of degree $\leq s$.

Pf: g entire $\Rightarrow$ $g(z) = \sum_{n=0}^{\infty} b_n z^n$, $\forall\ z \in \mathbb{C}$

By Cauchy integral famula (Fourier coefficients),

we have

$$\frac{1}{2\pi}\int_0^{2\pi} g(re^{i\theta})\, e^{-in\theta}\, d\theta = \begin{cases} b_n r^n , & n \geq 0 \\ 0 , & n < 0 \end{cases}$$

$$\Rightarrow \text{For } n>0, \quad \frac{1}{2\pi}\int_0^{2\pi} \overline{g(re^{i\theta})}\, e^{-in\theta}\, d\theta = 0$$

Hence $\frac{1}{2\pi}\int_0^{2\pi} (g+\bar{g})(re^{i\theta})\, e^{-in\theta}\, d\theta = b_n r^n$, $n>0$

ie. $\int_0^{2\pi} [\mathrm{Re}\, g(re^{i\theta})] \cdot e^{-in\theta}\, d\theta = \pi b_n r^n$, $\forall n>0$

For $n=0$, $\int_0^{2\pi} \mathrm{Re}\, g(re^{i\theta})\, d\theta = 2\pi\, \mathrm{Re}(b_0)$.

Note that $\int_0^{2\pi} e^{-in\theta}\, d\theta = 0$, $\forall\ n>0$,

we have $b_n = \frac{1}{\pi r^n}\int_0^{2\pi} [\mathrm{Re}\, g(re^{i\theta}) - Cr^s] e^{-in\theta}\, d\theta$

$\Rightarrow$ for $r = r_m$,

$$|b_n| \leq \frac{1}{\pi r_m^n}\int_0^{2\pi} [Cr_m^s - \mathrm{Re}\, g(r_m e^{i\theta})]\, d\theta$$

$$= \frac{2C}{r_m^{n-s}} - \frac{2\mathrm{Re}(b_0)}{r_m^n} \longrightarrow 0 \quad \text{as } r_m \to +\infty \quad \text{if } n>s$$

$\therefore$ g = poly. of degree $\leq s$. #

Pf of Hadamard's Theorem

Let $\quad E(z) = z^m \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)$

Lemma 4.2 $\Rightarrow \quad \left|1 - E_k\left(\frac{z}{a_n}\right)\right| \le c\left|\frac{z}{a_n}\right|^{k+1}$ for some c

Thm 2.1 $\Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{|a_n|^{k+1}} < +\infty \qquad$ since $k+1 > \rho_f$

we have $\quad \sum_{n=1}^{\infty} \left|1 - E_k\left(\frac{z}{a_n}\right)\right| \le C|z|^{k+1} \qquad C > 0$ indep. of z.

Hence the infinite product converges uniformly on $\{|z| \le R\}$, $\forall R > 0$,

this implies $E(z)$ is a well-defined entire function.

Since $E(z)$ has the same zeros as $f(z)$,

$\frac{f(z)}{E(z)}$ is holomorphic and nowhere vanishing.

$\Rightarrow \quad \frac{f(z)}{E(z)} = e^{g(z)}$ for some entire $g(z)$.

By Cor 5.4. for $|z| = r_m$,

$$e^{\operatorname{Re} g(z)} = \left|\frac{f(z)}{E(z)}\right| \le \frac{Ae^{B|z|^s}}{e^{-c|z|^s}} \qquad \forall s > \rho_f$$

$$= Ae^{(B+C)|z|^s}$$

$\Rightarrow \forall |z| = r_m, \quad \operatorname{Re} g(z) \le C'|z|^s$. (diff. C.)

By Lemma 5.5, $g(z) =$ polynomial of degree $\leq s$.

$\Rightarrow$ $g(z) =$ polynomial of degree $\leq k$.

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