Prop 3.2 Suppose $\{F_n(z)\}$ is a seq. of holo. functions on Ω (open).

If $\exists\ C_n > 0$ such that

$$\begin{cases} \sum C_n < \infty & \text{and} \\ |F_n(z) - 1| \leq C_n, & \forall z \in \Omega, \end{cases}$$

then

(i) $\prod_{n=1}^{\infty} F_n(z)$ converges uniformly in Ω to a holo. function $F(z)$.

(ii) If $F_n(z) \neq 0$, $\forall z \in \Omega$, $\forall n$, then

$$\frac{F'(z)}{F(z)} = \sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)}.$$

Pf: Write $F_n(z) = 1 + a_n(z)$.

Then by assumption $|a_n(z)| \leq C_n$

and hence $\sum a_n(z)$ uniformly absolute converges on Ω.

By the same argument, as $N \to +\infty$,

$$G_N(z) = \prod_{n=1}^{N} F_n(z) \longrightarrow F(z) = e^{\sum_{n=1}^{\infty} \log(1 + a_n(z))} \quad \text{(uniformly)}$$

which has to be holomorphic on Ω. This proves (i).

For (ii), (Thm 5.3 of Ch 2)

$G_N \to F$ uniformly $\Rightarrow$

$$G_N' \to F' \text{ uniformly on any cpt subset } K \subset \Omega$$

By Prop 3.1, the limit $F(z) \neq 0$, $\forall z \in \Omega$.

Hence $\forall$ cpt. subset $K \subset \Omega$, $\exists \delta > 0$ s.t. $|G_N(z)| \geq \delta$. (for N large)

$$\therefore \quad \sum_{n=1}^{N} \frac{F_n'(z)}{F_n(z)} = \frac{G_N'(z)}{G_N(z)} \to \frac{F'(z)}{F(z)} \text{ uniformly on } K.$$

Since $K \subset \Omega$ is arbitrary, we have $\dfrac{F'(z)}{F(z)} = \displaystyle\sum_{n=1}^{\infty} \frac{F_n'(z)}{F_n(z)}$.

#

3.2 <u>Example: the product formula for the sine function</u>

$$\boxed{\frac{\sin \pi z}{\pi} = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \qquad \text{—————} (3)}$$

We'll prove it by showing that

$$\boxed{\pi \cot \pi z = \lim_{N \to +\infty} \sum_{|n| \leq N} \frac{1}{z+n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \qquad \text{—————} (4)}$$

<u>Remarks</u>: (i) Formula (4) holds for $z \in \mathbb{C} \setminus \mathbb{Z}$ only

(ii) $\lim_{N\to+\infty} \sum_{|n|\le N} \frac{1}{z+n}$ is the principal value of $\sum_{n=-\infty}^{\infty} \frac{1}{z+n}$,

other arrangement may not converges.

Pf of (3) by (4).

Write $G(z) = \frac{\sin \pi z}{\pi}$

$$P(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

$P(z)$ is well-defined since $\left|\frac{-z^2}{n^2}\right| = \frac{|z|^2}{n^2} \le \frac{R^2}{n^2}$, $\forall z \in \{|z|<R\}$

Prop 3.2 $\Rightarrow$

$\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$ and hence $P(z)$ is well-defined on $\{|z|<R\}$.

Since $R>0$ is arbitary, $P(z)$ is entire.

Again by Prop 3.2, for $z \in \mathbb{C} \setminus \mathbb{Z}$,

$$\frac{P'(z)}{P(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2} = \pi \cot \pi z \quad \text{by formula (4)}.$$

Hence for $z \in \mathbb{C} \setminus \mathbb{Z}$

$$\left(\frac{P(z)}{G(z)}\right)' = \frac{P(z)}{G(z)} \left[\frac{P'(z)}{P(z)} - \frac{G'(z)}{G(z)}\right]$$

$$= \frac{P(z)}{G(z)} \left[\pi \cot \pi z - \frac{\cos \pi z}{\left(\frac{\sin \pi z}{\pi}\right)}\right] = 0$$

Since $\mathbb{C}\setminus\mathbb{Z}$ is connected, $P(z) = cG(z)$ for some constant c.
(and clearly extends to whole $\mathbb{C}$)

Letting $z \to 0$ in $\dfrac{P(z)}{z} = c\,\dfrac{G(z)}{z}$ (near, but $\neq 0$),

i.e. $\displaystyle\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right) = c\,\frac{\sin \pi z}{\pi z}$, we have $c = 1$. #

Pf of formula (4)

Let $F(z) = \pi \cot \pi z$.

Then (i) $F(z+1) = F(z)$, $z \in \mathbb{C}\setminus\mathbb{Z}$

(ii) $F(z) = \frac{1}{z} + F_0(z)$, where F_0 analytic near 0.

(iii) $z = n \in \mathbb{Z}$ are simple pole of $F(z)$, &

$F(z)$ has no other singularities.

Note that

$$G(z) = \lim_{N\to+\infty} \sum_{|n|\le N} \frac{1}{z+n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}$$ (not the G in previous step)

clearly satisfies (ii). (in fact, holo in $|\operatorname{Im} z| < 1$)

And $\displaystyle G(z+1) = \lim_{N\to+\infty} \sum_{|n|\le N} \frac{1}{z+1+n}$

$$= \lim_{N\to+\infty}\left[\frac{1}{z+1-N} + \frac{1}{z-N+2} + \cdots + \frac{1}{z+N} + \frac{1}{z+1+N}\right]$$

$$= \lim_{N\to+\infty} \left[\left(\sum_{|n|\le N} \frac{1}{z+n} \right) - \frac{1}{z-N} + \frac{1}{z+1+N} \right]$$

$$= G(z) \quad \text{as} \quad \lim_{N\to+\infty} \frac{1}{z+1+N} = 0 = \lim_{N\to+\infty} \frac{1}{z-N}.$$

This proves G also satisfies (i).

Then (i) and (ii) together implies (iii).

Now consider $\Delta(z) = F(z) - G(z)$.

Then by (i), $\Delta(z+1) = \Delta(z)$ (periodic)

By (ii) $\Delta(z) = \frac{1}{z} + F_0(z) - \frac{1}{z} - G_0(z)$ near $z=0$

$\left(\text{where } G_0(z) = \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}\right)$

$= F_0(z) - G_0(z)$ analytic near $z=0$

$\therefore$ $z=0$ is a removable singularity of $\Delta(z)$.

Together with (i) and (iii), all $z=n$ are removable singularities and hence $\Delta(z)$ is entire.

If $z = x+iy$ with $|x| \le \frac{1}{2}$ and $|y| > 1$,

then $$\cot \pi z = i \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} = i \frac{e^{-\pi y + i\pi x} + e^{\pi y - i\pi x}}{e^{-\pi y + i\pi x} - e^{\pi y - i\pi x}}$$

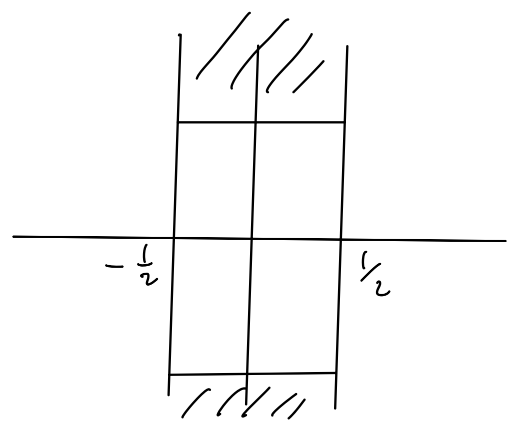
$$= i \frac{e^{-2\pi y} + e^{-2i\pi x}}{e^{-2\pi y} - e^{-2i\pi x}}$$

$$\therefore \quad |\cot \pi z|^2 = \frac{(e^{-2\pi y} + \cos 2\pi x)^2 + (\sin 2\pi x)^2}{(e^{-2\pi y} - \cos 2\pi x)^2 + (\sin 2\pi x)^2}$$

$$\leq \frac{(e^{-2\pi y}+1)^2}{(e^{-2\pi y}-1)^2} \qquad (Ex!)$$

$$\Rightarrow \quad |\cot \pi z| \leq \frac{1+e^{-2\pi}}{1-e^{-2\pi}} \quad \text{for all } |y|>1 \ \& \ |x|\leq \tfrac{1}{2}$$

$\therefore |F(z)| \leq C$ on $\{ |\mathrm{Re}\, z| \leq \frac{1}{2}, \ |\mathrm{Im}\, z| > 1 \}$



for some C.

Now for $G(z)$,

$$G(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}$$

$\Rightarrow$ On $\{ z = x+iy : |x| \leq \frac{1}{2} \ \& \ |y| > 1 \}$,

$$|G(z)| \leq \frac{1}{|z|} + \sum_{n=1}^{\infty} \left| \frac{2z}{z^2-n^2} \right|$$

$$\leq 1 + C \sum_{n=1}^{\infty} \frac{|y|}{y^2+n^2} \qquad (Ex!)$$

By Riemann sum
$$\sum_{n=1}^{\infty} \frac{|y|}{y^2+n^2} \leq \int_0^{\infty} \frac{|y|}{y^2+x^2}\,dx$$

$$= \int_0^{\infty} \frac{|y|}{y^2+y^2t^2}\,|y|\,dt$$

$$= \int_0^{\infty} \frac{dt}{1+t^2} = B.$$

$\Rightarrow \quad |G(z)|$ is also bounded on $\{z=x+iy : |x|\leq \frac{1}{2} \ \& \ |y|>1\}$.

$\therefore \quad |\Delta(z)|$ is bounded on $\{z=x+iy : |x|\leq \frac{1}{2} \ \& \ |y|>1\}$.

Since $\Delta(z)$ is entire, it is bounded on

$$\{z=x+iy : |x|\leq \tfrac{1}{2} \ \& \ |y|\leq 1\}$$

Together we have $|\Delta(z)|$ is bounded on $\{z=x+iy : |x|\leq \frac{1}{2}\}$.

Then by periodicity $\Delta(z+1)=\Delta(z)$, we conclude that $\Delta(z)$ is bounded on $\mathbb{C}$.

Hence Liouville's Thm $\Rightarrow$ $\Delta(z)=$ constant $=c$

Finally $c=\Delta(-z) = F(-z)-G(-z)$

$$= \pi\cot(-\pi z) - \left[\frac{1}{-z} + \sum_{n=1}^{\infty} \frac{2(-z)}{(-z)^2-n^2}\right]$$

$$= -\left[\pi\cot\pi z - \left(\frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}\right)\right]$$

$$= -\Delta(z) = -c$$

$\therefore \ c=0$

$$\Rightarrow \quad \pi\cot\pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2} \quad \#$$