§3 Paley–Wiener Theorem

Omitted except the following theorem

Thm 3.4 (Phragmén – Lindelöf)

Suppose • F is holo on $S = \{z : -\frac{\pi}{4} < \arg z < \frac{\pi}{4}\}$ and continuous on $\overline{S}$ (closure).

• $|F(z)| \leq 1$ for $z \in \partial S$ (ie $|\arg z| = \frac{\pi}{4}$)

If $\exists$ constants $C_1, C_2 > 0$ such that

$$|F(z)| \leq C_1 e^{C_2|z|}, \ \forall z \in S,$$

then $|F(z)| \leq 1, \ \forall z \in S.$

Remark: This is a "version" of maximum principle, but on unbounded domain.

$$\sup_{\overline{S}} |F(z)| = \sup_{\partial S} |F(z)|$$

which is usually not true without the growth condition.

eg • $G(z) = e^{z^2}$ is holo on S

• $|G(re^{\pm i\frac{\pi}{4}})| = |e^{r^2 e^{\pm i\frac{\pi}{2}}}| = |e^{\pm r^2 i}| = 1,$

but $|G(x)| = e^{x^2} \to +\infty$ as $x \to +\infty$

$\therefore$ $G(z)$ is unbounded on S.

Pf of Thm 3.4

$\forall \varepsilon > 0$, consider $F_\varepsilon(z) = F(z) e^{-\varepsilon z^{\frac{3}{2}}}$ for $z \in S$.

Note that $z \in S \Rightarrow z = re^{i\theta}$ with $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$.

$$-\frac{3\pi}{8} < \frac{3\theta}{2} < \frac{3\pi}{8}$$

$\Rightarrow \cos\frac{3\theta}{2} \geq \delta > 0$ for some δ.

Hence

$$|e^{-\varepsilon z^{\frac{3}{2}}}| = e^{-\varepsilon r^{\frac{3}{2}} \cos(\frac{3\theta}{2})} \leq e^{-\varepsilon \delta r^{\frac{3}{2}}} \leq 1$$

Therefore, the growth condition on $F(z)$ implies

$$|F_\varepsilon(z)| = |F(z)||e^{-\varepsilon z^{\frac{3}{2}}}| \leq c_1 e^{c_2 r} e^{-\varepsilon \delta r^{\frac{3}{2}}}$$

$$= c_1 e^{-(\varepsilon\delta - c_2 r^{-\frac{1}{2}}) r^{3/2}}$$

For $r \gg 1$, $\varepsilon\delta - c_2 r^{-\frac{1}{2}} > 0$, hence

$F_\varepsilon(z)$ is rapidly decreasing.

In particular, F_ε is bounded on $\overline{S}$.

Let $M_\varepsilon = \sup_{z \in \overline{S}} |F_\varepsilon(z)|$.

If $F_\varepsilon \equiv 0$, then $F \equiv 0$, we are done.

If $F_\varepsilon \not\equiv 0$, then $\exists\, w_j \in S$, $j = 1, 2, \cdots$, s.t.

$$|F_\varepsilon(w_j)| \to M_\varepsilon \quad \text{as } j \to +\infty$$

and $\quad M_\varepsilon > 0$.

Since $|F_\varepsilon| \to 0$ as $|z| \to +\infty$, we conclude that

$\{w_j\}$ is bounded.

Therefore $\exists\, w \in \bar{S}$ s.t. $w_j \to w$. (by passing to subseq.)

By maximum principle (Thm 4.5), w can't be an interior point of S. Hence $w \in \partial S$.

Continuity of F on $\bar{S}$ and $|F| \leq 1$ on ∂S

implies (check!)

$$M_\varepsilon = |F_\varepsilon(w)| \leq |F(w)|\,|e^{-\varepsilon w^{\frac{3}{2}}}| \leq 1.$$

ie.

$$\left|F(z)\, e^{-\varepsilon z^{\frac{3}{2}}}\right| \leq 1, \quad \forall z \in S$$

$$\Rightarrow \quad |F(z)| \leq e^{\varepsilon |z|^{\frac{3}{2}}}, \quad \forall z \in S$$

Since $\varepsilon > 0$ is arbitary, $|F(z)| \leq 1$, $\forall z \in S$.

\#

Ch5 Entire Functions

§1 Jensen's Formula

In this section, $D_R = \{z : |z| < R\}$ $(R > 0)$

$C_R = \{z : |z| = R\} = \partial D_R$

Thm 1.1 (Jensen's Formula)

Let
- Ω = open set s.t. $\overline{D_R} \subset \Omega$. (hence $0 \in \Omega$)
- f holo. on Ω,
- $f(z) \neq 0$ for $z = 0$ or $z \in C_R$
- $z_1, \cdots, z_N \in D_R$ are (all) the zeros of f in D_R

  (i.e. $z_1, \cdots, z_N \notin C_R$) (countable multiplicity)

Then

$$(1) \qquad \log|f(0)| = \sum_{k=1}^{N} \log \frac{|z_k|}{R} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta$$

Pf: (My Steps are different from the Text)

Step 1 If g holo on $\overline{D_R}$ and $g(z) \neq 0$, $\forall z \in \overline{D_R}$, then

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\theta})|\, d\theta.$$

Pf: g holo on $\overline{D}_R \Rightarrow g$ holo on $D_{R+\varepsilon}$ for some $\varepsilon>0$.

Since $D_{R+\varepsilon}$ is simply connected & $g(z)\neq 0$ on $D_{R+\varepsilon}$,

there exists a holo. function $h(z)$ on $D_{R+\varepsilon}$ s.t.

$$g(z)=e^{h(z)}. \quad \text{(Thm 6.2 in Ch3 of Text)}$$

$$\Rightarrow \quad |g(z)| = |e^{h(z)}| = e^{\mathrm{Re}\, h(z)}$$

By mean value property (of harmonic functions),

(Cor 7.3 in Ch3 of Text)

$$\frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\theta})|\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\, h(Re^{i\theta})\,d\theta$$

$$= \mathrm{Re}\, h(0)$$

$$= \log|g(0)|. \quad ※$$

<u>Step 2</u> $\int_0^{2\pi} \log|1-ae^{i\theta}|\,d\theta = 0, \quad \forall\, |a|<1.$

Pf: consider $F(z)=1-az$ on $\mathbb{D}=\{|z|<1\}$

Then • $F(z)$ is holo. on $\overline{\mathbb{D}}$,

• $F(z)\neq 0$ on $\overline{\mathbb{D}}$, since $|a|<1$

By Step 1,

$$0 = \log|F(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|F(e^{i\theta})|\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log|1-ae^{i\theta}|\, d\theta \quad \#$$

<u>Step3</u> General case.

<u>Pf</u>: By assumption & Thm 1.1 of Ch3,

$f(z) = (z-z_1)\cdots(z-z_N)\, g(z)$ for some holo. function

g on Ω s.t. $g(z) \neq 0$, $\forall z \in \overline{D_R}$.

Then $\log|f(0)| = \log|z_1\cdots z_N||g(0)|$

$$= \sum_{k=1}^{N} \log|z_k| + \log|g(0)|$$

(By Step 1) $$= \sum_{k=1}^{N} \log|z_k| + \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\theta})|\, d\theta.$$

$(z_k \notin C_R)$ $$= \sum_{k=1}^{N} \log|z_k| + \frac{1}{2\pi}\int_0^{2\pi} \log \frac{|f(Re^{i\theta})|}{|Re^{i\theta}-z_1|\cdots|Re^{i\theta}-z_N|}\, d\theta$$

$$= \sum_{k=1}^{N} \log|z_k| + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\, d\theta$$

$$- \sum_{k=1}^{N} \frac{1}{2\pi}\int_0^{2\pi} \log R\left|1-\frac{z_k}{R}e^{-i\theta}\right| d\theta$$

(by change of variable in last term $\theta \mapsto -\theta$)

$$= \sum_{k=1}^{N} \log \frac{|z_k|}{R} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| d\theta + \frac{1}{2\pi}\sum_{k=1}^{N}\int_0^{2\pi} \log\left|1 - \frac{z_k}{R}e^{i\theta}\right| d\theta$$

$\left(\frac{|z_k|}{R} < 1 \text{ \& Step 2}\right)$

$$= \sum_{k=1}^{N} \log \frac{|z_k|}{R} + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| d\theta$$

//

Def Notations as in Thm 1.1, we define the function of $r \in (0,R)$

$$n_f(r) = \text{number of zeros of } f \text{ in } D_r$$

(or simply $n(r)$) (counting multiplicity)

Remark: $r_1 > r_2 \Rightarrow n(r_1) \geq n(r_2)$ (nondecreasing)

Lemma 1.2 If f holo on $\overline{D}_R$ & $f(0) \neq 0$.

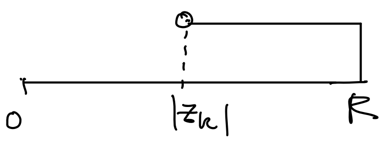
If $z_1, \cdots, z_N$ are the zeros of f in D_R, then

$$\int_0^R n(r)\frac{dr}{r} = \sum_{k=1}^{N} \log\left|\frac{R}{z_k}\right|$$

Pf: Clearly $\displaystyle\sum_{k=1}^{N} \log\left|\frac{R}{z_k}\right| = \sum_{k=1}^{N} \int_{|z_k|}^{R} \frac{dr}{r}$

Define the characteristic function

$$\eta_k(r) = \begin{cases} 1, & r > |z_k| \quad (r<R) \\ 0, & r \le |z_k| \end{cases}$$



Then $\displaystyle\sum_{k=1}^{N} \log\frac{R}{|z_k|} = \sum_{k=1}^{N} \int_0^R \eta_k(r)\,\frac{dr}{r}$

$$= \int_0^R \left(\sum_{k=1}^{N} \eta_k(r)\right) \frac{dr}{r}$$

Note that $\displaystyle\sum_{k=1}^{N} \eta_k(r) = \underbrace{1+\cdots+1}_{\text{those } k \text{ s.t. } r>|z_k|,\ (\text{i.e. } z_k \in D_r)} + 0 + \cdots + 0 = \pi(r)$,

we've proved the Lemma ※.

By the Lemma 1.2, the Jensen's formula can be rewritten as

$$(2) \quad \int_0^R \pi(r)\frac{dr}{r} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta - \log|f(0)|$$

for f holo. on $\overline{D_R}$ with $f(0)\neq 0$ & $f(z)\neq 0$ $\forall z\in C_R$

§2 Functions of Finite Order

Def: Let f be an entire function. If $\exists \rho > 0$ s.t. for some constants $A, B > 0$,

$$|f(z)| \leq A e^{B|z|^{\rho}} \quad \forall z \in \mathbb{C},$$

then we say that f has an order of growth $\leq \rho$.

And define the order of growth of f as

$$\rho_f = \inf\{\rho : f \text{ has an order of growth} \leq \rho\},$$

eg: The order of growth of e^{z^2} is 2. (Ex!)

Remarks:

- Clearly, if f has an order of growth $\leq \rho_1$ and $\rho_1 < \rho_2$, then f has an order of growth $\leq \rho_2$. (Ex!)

- It is easy to see that for $f(z) = e^{z^2}$, ($\rho_f = 2$) $\exists A, B > 0$ s.t.

$$|f(z)| \leq A e^{B|z|^{\rho_f}}, \quad \forall z \in \mathbb{C}.$$

But, in general, the definition of ρ_f only implies $\forall \varepsilon > 0$, $\exists A, B > 0$ s.t.

$$|f(z)| \leq A e^{B|z|^{\rho_f + \varepsilon}}, \quad \forall z \in \mathbb{C}.$$

Thm 2.1 If f is an entire function and has an order of growth $\leq \rho$, then

(i) $\pi(r) \leq C r^{\rho}$ for some $C>0$ & sufficiently large r.

(ii) If $z_1, z_2, \cdots$ are the zeros of f with $z_k \neq 0$,

then $\forall\, s > \rho$ we have

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^s} < \infty .$$

Pf: If $f(0)=0$, then

$$F(z) = \frac{f(z)}{z^l}, \quad \text{where } l = \text{order of zero at } 0,$$

is an entire function & $F(0) \neq 0$.

Then the assumption $\Rightarrow$

$$|F(z)| = \frac{|f(z)|}{|z|^l} \quad \text{is bounded in } \{|z| \leq 1\} \text{ and}$$

$$|F(z)| \leq |f(z)| \leq A e^{B|z|^{\rho}} \quad \text{for } \{|z|>1\}$$

Hence F also has an order of growth $\leq \rho$, with the same zeros $z_1, z_2, \cdots, z_k \neq 0$, as f.

And
$$\pi_f(r) = \pi_F(r) + l.$$

Therefore, we only need to show Thm 2.1 for entire function f with $f(0) \neq 0$.

If $f(0) \neq 0$, then we can apply formula (2) in the previous section:

$$\int_0^R \pi(r)\frac{dr}{r} = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta - \log|f(0)|$$

Take $R = 2r$, we have

$$\int_r^{2r} \pi(t)\frac{dt}{t} \leq \int_0^{2r} \pi(t)\frac{dt}{t}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta - \log|f(0)|$$

Since π is non-decreasing, $\pi(t) \geq \pi(r)$ $\forall\ t \in (r, 2r)$.

$$\therefore \quad \int_r^{2r} \pi(t)\frac{dt}{t} \geq \pi(r)\int_r^{2r}\frac{dt}{t} = \pi(r)\log 2.$$

$$\Rightarrow \quad \pi(r)\log 2 \leq \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|\,d\theta - \log|f(0)|$$

$$\leq \log\left[A e^{B(2r)^\rho}\right] - \log|f(0)|$$

$$= (2^\rho B)\,r^\rho + \log\frac{A}{|f(0)|}$$

$\leq C r^\rho$ for r sufficiently large. (Ex!)

for some $C > 0$

This proves part (i).

(so one can see that in general, C and how large r needed depends on the function f and ρ.)

To prove part (ii), we note that there is only finitely many zeros of f inside $\{|z_k|<1\}$ and $\{2^j \le |z_k| < 2^{j+1}\}$.

Then
$$\sum_{1\le|z_k|<2^{N+1}} \frac{1}{|z_k|^s} = \sum_{j=0}^{N}\left(\sum_{2^j\le|z_k|<2^{j+1}} \frac{1}{|z_k|^s}\right)$$

$$\le \sum_{j=0}^{N} \frac{1}{2^{js}} \#\{z_k : 2^j \le |z_k| < 2^{j+1}\}$$

$$\le \sum_{j=0}^{N} \frac{1}{2^{js}} n(2^{j+1})$$

by part (i)
$$\le C\sum_{j=0}^{N} \frac{1}{2^{js}} 2^{(j+1)\rho}$$

$$= 2^\rho C \sum_{j=0}^{N} \left(\frac{1}{2^{s-\rho}}\right)^j$$

(since $s>\rho$)
$$< 2^\rho C \sum_{j=0}^{\infty} \left(\frac{1}{2^{s-\rho}}\right)^j < \infty$$

Letting $N\to+\infty$ & using $\#\{|z_k|<1\}$ finite $\Rightarrow$

$$\sum_{k=1}^{\infty} \frac{1}{|z_k|^s} < \infty \quad \#$$

(used absolutely convergence and hence the series can be rearranged)

Note: $s>\rho$ is important in the proof.

(Can't be improved to $s=\rho$.)

eg 1 Let $f(z) = \sin \pi z = \frac{e^{i\pi z} - e^{-i\pi z}}{2}$.

Then $|f(z)| \leq e^{\pi|z|}$, $\forall z \in \mathbb{C}$. (Ex!)

ie. f has an order of growth ≤ 1.

On the other hand, if $\exists \rho > 0$, $A, B > 0$ s.t.

$$|f(z)| \leq A e^{B|z|^\rho}, \quad \forall z \in \mathbb{C}.$$

Then
$$\left|\frac{e^{-\pi y} - e^{\pi y}}{2}\right| = |f(iy)| \leq A e^{B|y|^\rho}$$

$$\Rightarrow \quad 1 - e^{-2\pi y} \leq 2A e^{(By^\rho - \pi y)} \quad \text{for } y > 0$$

If $\rho < 1$, we have $1 \leq 2A \lim_{y \to +\infty} e^{By^\rho - \pi y} = 0$

which is a contradiction.

$\therefore \quad \rho_f = \inf \rho = 1$.

Note that the zeros are $n \in \mathbb{Z}$, the Thm 2.1 $\Rightarrow$

$$\sum_{n \neq 0} \frac{1}{|n|^s} < \infty \quad \text{for } s > 1.$$

But $\sum_{n \neq 0} \frac{1}{|n|^s}$ diverges for $s \leq 1$

$\#$

eg 2 $\quad f(z) = \cos z^{\frac{1}{2}} = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{(2n)!}$

Then $\quad \rho_f = \frac{1}{2} \quad$ (Ex!)

f has zeros at $z_n = \left[(n+\frac{1}{2})\pi\right]^2$, and

$$\sum_{n\in\mathbb{Z}} \frac{1}{|z_n|^s} = \sum_{n\in\mathbb{Z}} \frac{1}{\left[(n+\frac{1}{2})\pi\right]^{2s}} \begin{cases} \text{converges if } s > \frac{1}{2} \\ \text{diverge otherwise.} \end{cases}$$

§3 Infinite Products

3.1 Generalities

Def Given $\{a_n\}_{n=1}^{\infty}$ ($a_n \in \mathbb{C}$), we say that the infinite product (or just product)

$$\prod_{n=1}^{\infty} (1+a_n) \quad \underline{\text{converges}}$$

if $\displaystyle\lim_{N\to\infty} \prod_{n=1}^{N} (1+a_n)$ exists.

Remark: $\displaystyle\prod_{n=1}^{N} (1+a_n)$ is called the N-term partial product

Prop 3.1: $\displaystyle\sum |a_n| < \infty \Rightarrow \prod_{n=1}^{\infty} (1+a_n)$ converges

In this case, $\displaystyle\prod_{n=1}^{\infty} (1+a_n) = 0 \Leftrightarrow \exists\, n_0$ s.t. $1+a_{n_0} = 0$

Pf: $\sum |a_n| < \infty \Rightarrow |a_n| < \frac{1}{2}$ for sufficiently large n

$\Rightarrow$ for suff. large n,

$$\log(1+a_n) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{a_n^k}{k} \quad \text{is well-defined and satisfies}$$

$$1+a_n = e^{\log(1+a_n)} \qquad (\text{actually holds } \forall\, |a_n|<1)$$

Hence $\prod_{n=1}^{N}(1+a_n) = \prod_{n=1}^{N} e^{\log(1+a_n)} = e^{\sum_{n=1}^{N}\log(1+a_n)}$.

By the definition of $\log(1+a_n)$, we have for sufficiently large n,

$$|\log(1+a_n)| \leq 2|a_n| \quad \text{for } |a_n|<\frac{1}{2}$$

$$\Rightarrow \quad \sum_{n=1}^{N}|\log(1+a_n)| \leq 2\sum_{n=1}^{N}|a_n|$$

$\sum|a_n|<\infty \Rightarrow \sum_{n=1}^{N}\log(1+a_n)$ converges absolutely

$$\therefore \quad \lim_{N\to+\infty}\prod_{n=1}^{N}(1+a_n) = e^{\sum_{n=1}^{\infty}\log(1+a_n)} \text{ exists.}$$

In this case, if $\exists n_0$ s.t. $1+a_{n_0}=0$, then

$\lim_{N\to+\infty}\prod_{n>n_0}^{N}(1+a_n)$ exists and

$$\lim_{N\to\infty}\prod_{n=1}^{N}(1+a_n) = \prod_{n=1}^{n_0}(1+a_n)\cdot\lim_{N\to\infty}\prod_{n>n_0}^{N}(1+a_n) = 0$$

Since $\sum|a_n|<\infty$, if $1+a_n\neq 0, \forall n$.

Then $$\lim_{N\to\infty}\prod_{n=1}^{N}(1+a_n) = e^{\sum_{n=1}^{\infty}\log(1+a_n)} \neq 0. \quad \#$$