Thm 2.4 (Poisson Summation Formula)

If $f \in \mathfrak{F}$, then $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$

Pf: $f \in \mathfrak{F} \Rightarrow f \in \mathfrak{F}_a$ for some $a>0$.

$\Rightarrow f$ holo. on $S_a = \{x+iy : |y|<a\}$

Consider $g(z) = \dfrac{f(z)}{e^{2\pi i z}-1}$ on S_a.

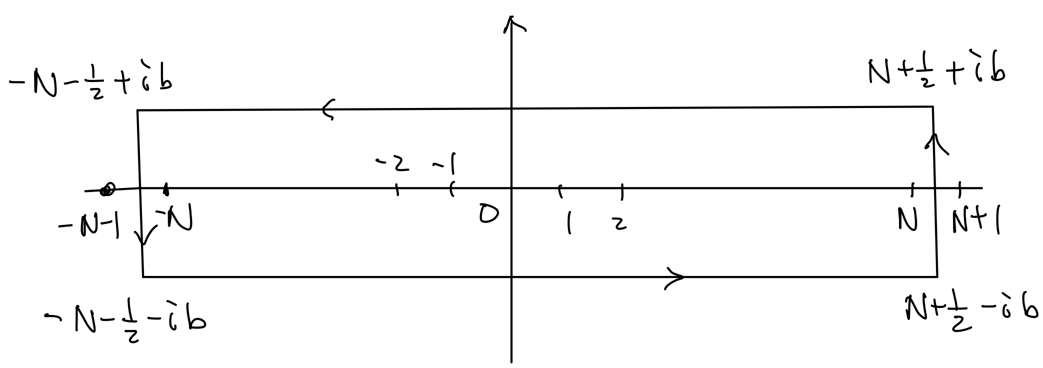
It is easy to see $\dfrac{1}{e^{2\pi i z}-1}$ has simple pole at $n\in\mathbb{Z}$

with $\mathrm{res}_n \dfrac{1}{e^{2\pi i z}-1} = \dfrac{1}{2\pi i}$ (Ex!)

Hence $g(z) = \dfrac{f(z)}{e^{2\pi i z}-1}$ has simple pole at $n\in\mathbb{Z}$

with $\mathrm{res}_n g = \dfrac{f(n)}{2\pi i}$ (except $f(n)=0$, where g has a removable singularity $\Rightarrow$ no contribution to the contour integral.)

Applying Residue Formula (Cor 2.3 of Ch 3 of Text) to the contour γ_N, $N\in\mathbb{Z}^+$, as in the figure, for $0<b<a$,



we have

$$2\pi i \sum_{|n|\leq N} \operatorname{res}_n g = \int_{\gamma_N} g(z)\,dz$$

i.e.
$$\sum_{|n|\leq N} f(n) = \int_{\gamma_N} \frac{f(z)}{e^{2\pi i z}-1}\,dz.$$

Note that $f \in \mathfrak{F}_a \Rightarrow \exists A>0$ s.t. $|f(z)| \leq \dfrac{A}{1+|\operatorname{Re}(z)|^2}$

$$\Rightarrow |f(n)| \leq \frac{A}{1+n^2} \quad \forall n \in \mathbb{Z}$$

$$\therefore \sum_{|n|\leq N} f(n) \to \sum_{n\in\mathbb{Z}} f(n) \quad \text{as } N \to +\infty.$$

And
$$\left| \int_{\pm(N+\frac{1}{2})-ib}^{\pm(N+\frac{1}{2})+ib} \frac{f(z)}{e^{2\pi i z}-1}\,dz \right| \leq \frac{C}{N^2} \quad (N \in \mathbb{Z}^+) \qquad (\text{Ex!})$$

for some constant C depending on A and b only

Hence letting $N \to +\infty$ in $\displaystyle\sum_{|n|\leq N} f(n) = \int_{\gamma_N} \frac{f(z)}{e^{2\pi i z}-1}\,dz$,

we have
$$\sum_{n\in\mathbb{Z}} f(n) = \int_{L_1} \frac{f(z)}{e^{2\pi i z}-1}\,dz - \int_{L_2} \frac{f(z)}{e^{2\pi i z}-1}\,dz$$

where
$$\begin{cases} L_1 = \{x+iy : y=-b\} \text{ oriented left to right} \\ L_2 = \{x+iy : y=b\} \text{ oriented left to right.} \end{cases}$$

Note that on L_1, $|e^{2\pi i z}| = |e^{2\pi i (x-ib)}| = e^{2\pi b} > 1$

$$\therefore \quad \frac{1}{e^{2\pi i z}-1} = \frac{1}{e^{2\pi i z}} \cdot \frac{1}{1-e^{-2\pi i z}}$$

$$= e^{-2\pi i z} \sum_{k=0}^{\infty} e^{-2\pi i k z}$$

Similarly on L_2, $|e^{2\pi i z}| = e^{-2\pi b} < 1$

$$\frac{1}{e^{2\pi i z}-1} = -\sum_{k=0}^{\infty} e^{2\pi i k z}$$

$$\therefore \quad \sum_{n\in\mathbb{Z}} f(n) = \int_{L_1} f(z)\, e^{-2\pi i z} \sum_{k=0}^{\infty} e^{-2\pi i k z}\, dz$$

$$+ \int_{L_2} f(z) \sum_{k=0}^{\infty} e^{2\pi i k z}\, dz$$

Since $|f(z)| \leq \frac{A}{1+|\mathrm{Re}\, z|^2}$, both $\int_{L_1}$ & $\int_{L_2}$ can be interchanged with $\sum_{k=0}^{\infty}$, and we have

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k=0}^{\infty} \int_{L_1} f(z)\, e^{-2\pi i (k+1) z}\, dz$$

$$+ \sum_{k=0}^{\infty} \int_{L_2} f(z)\, e^{2\pi i k z}\, dz$$

Then using $(*)_1$ & $(*)_2$ in the proof of Thm 2.1, we have

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k=0}^{\infty} \hat{f}(k+1) + \sum_{k=0}^{\infty} \hat{f}(-k) = \sum_{k\in\mathbb{Z}} \hat{f}(k)$$

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Applications of Poisson summation formula

(1) For $t>0$, define the <u>theta function</u> by
$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}.$$
Then $\vartheta(t) = t^{-\frac{1}{2}} \vartheta(\frac{1}{t})$, $\forall\, t>0$.

Pf: This follows from a more general formula
$$\sum_{n=-\infty}^{\infty} e^{-\pi t(n+a)^2} = \sum_{n=-\infty}^{\infty} t^{-\frac{1}{2}} e^{-\pi\frac{n^2}{t}} e^{2\pi i n a} \quad \text{for } a\in\mathbb{R}.$$

To prove this, we observe that by Eg 1 of Ch 2,
$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\, dx = e^{-\pi\xi^2}$$
(i.e. Fourier transform of $e^{-\pi x^2}$ is $e^{-\pi\xi^2}$.)

Change of variable $x \mapsto \sqrt{t}(x+a)$ $\Rightarrow$
$$\int_{-\infty}^{\infty} e^{-\pi t(x+a)^2} e^{-2\pi i x(\sqrt{t}\xi)} e^{-2\pi i a(\sqrt{t}\xi)} \sqrt{t}\, dx = e^{-\frac{\pi}{t}(\sqrt{t}\xi)^2}$$
$$\Rightarrow \int_{-\infty}^{\infty} e^{-\pi t(x+a)^2} e^{-2\pi i x\zeta}\, dx = \frac{1}{\sqrt{t}} e^{-\frac{\pi}{t}\zeta^2} e^{2\pi i a\zeta} \qquad (\zeta = \sqrt{t}\xi)$$

i.e. $$\hat{f}(\zeta) = \frac{1}{\sqrt{t}} e^{-\frac{\pi}{t}\zeta^2} e^{2\pi i a\zeta}$$

for the function $f(x) = e^{-\pi t(x+a)^2}$.

Then Poisson summation famula $\Rightarrow$ (Check $f \in \mathfrak{F}$)

$$\sum_{n=-\infty}^{\infty} e^{-\pi t(n+a)^2} = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{t}} e^{-\frac{\pi}{t} n^2} e^{2\pi i n a}$$

which is the required famula.

Putting $a=0$, we have

$$\vartheta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi t n^2} = \frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{t}} = \frac{1}{\sqrt{t}} \vartheta\left(\frac{1}{t}\right)$$

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(2) $\forall a \in \mathbb{R}$, $t > 0$, we have

$$\sum_{n=-\infty}^{\infty} \frac{e^{-2\pi i a n}}{\cosh\left(\frac{\pi n}{t}\right)} = \sum_{n=-\infty}^{\infty} \frac{t}{\cosh(\pi(n+a)t)}$$

Pf: Eg 3 of Ch3 gives

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{\cosh(\pi x)} dx = \frac{1}{\cosh(\pi \xi)}$$

Consider

$$f(x) = \frac{e^{-2\pi i a x}}{\cosh\left(\pi \frac{x}{t}\right)}, \text{ then}$$

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i a x}}{\cosh\left(\pi \frac{x}{t}\right)} e^{-2\pi i x \xi} dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{-2\pi i a t x}}{\cosh(\pi x)} e^{-2\pi i t x \xi} \, t \, dx = \frac{t}{\cosh(\pi t(\xi + a))}$$

$\therefore$ Poisson summation formula $\Rightarrow$ (Check $f \in \mathfrak{F}$)

$$\sum_{n=-\infty}^{\infty} \frac{e^{-2\pi i a n}}{\cosh(\pi \frac{n}{t})} = \sum_{n=-\infty}^{\infty} \frac{t}{\cosh(\pi t(n+a))}$$

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