§5 Homotopies and Simply Connected Domains

Def: Ω open in $\mathbb{C}$; $\gamma_0(t)$ & $\gamma_1(t)$, $t\in[a,b]$, curves in Ω with common end points, i.e. $\gamma_0(a)=\gamma_1(a)=\alpha$; $\gamma_0(b)=\gamma_1(b)=\beta$.

γ_0 & γ_1 are said to be homotopic in Ω if $\exists$ continuous map

$H(s,t): [0,1]\times[a,b] \to \Omega$ such that

$$H(0,t)=\gamma_0(t) \ \& \ H(1,t)=\gamma_1(t), \quad \forall t\in[a,b].$$

and $$H(s,a)=\alpha \ \& \ H(s,b)=\beta, \quad \forall s\in[0,1]$$

Remark: Usually think of $H(s,t)=\gamma_s(t)$ as a family of curves in Ω with the same end points $\gamma_s(a)=\alpha$ & $\gamma_s(b)=\beta$

s.t. for $s=0$ & 1 they are the original two curves γ_0 & γ_1.

Hence "one curve can be deformed continuously into the other curve without ever leaving Ω".

Thm 5.1 If f holo. in Ω, then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

provided γ_0 and γ_1 are homotopic in Ω.

Def A region $\Omega \subset \mathbb{C}$ is simply connected if any two curves in Ω with the same end points are homotopic in Ω.

Thm 5.2 & Cor 5.3 If f is holomorphic in a simply connected domain Ω, then

(1) $\exists\, F: \Omega \to \mathbb{C}$ holo s.t. $F' = f$;

(2) $\int_\gamma f(z)\,dz = 0 \quad \forall$ closed curve γ in Ω.

§6 The Complex Logarithm

Thm 6.1 Suppose Ω is simply connected,

$$1 \in \Omega \text{ but } 0 \notin \Omega.$$

Then $\exists$ a branch of the logarithm $F(z) = \log_\Omega z$ s.t.

(i) F is holo. in Ω

(ii) $e^{F(z)} = z, \ \forall z \in \Omega$

(iii) $F(r) = \log r \quad \forall\, r \in \mathbb{R}$ and near to 1.

- Principal branch of the logarithm

$$\begin{cases} \Omega = \mathbb{C} \setminus (-\infty, 0], \\ \log z = \log r + i\theta \quad \text{with } \boxed{|\theta| < \pi} \quad \text{and } z = re^{i\theta}. \end{cases}$$

Thm 6.2 • Ω = simply connected region,

• $f: \Omega \to \mathbb{C}$ holo. & $f(z) \neq 0, \forall z \in \Omega$.

Then $\exists\, g: \Omega \to \mathbb{C}$ holo. s.t.

$$e^{g(z)} = f(z) \qquad \forall z \in \Omega.$$

(g is denoted by $\log f$)

§7 Fourier Series and Harmonic Functions

Thm 7.1 $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges in $D_R(z_0)$.

Then $\forall\, r \in (0, R)$,

$$\frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta}\, d\theta = \begin{cases} a_n r^n, & \text{for } n \geq 0 \\ 0, & \text{for } n < 0 \end{cases}$$

Remarks (i) This is just the Cauchy integral formula applies to the circle

$$\gamma(\theta) = z_0 + re^{i\theta}, \quad \theta \in [0, 2\pi]$$

(ii) The LHS is the Fourier coefficients (up to a const.)

of the 2π-periodic function $f(z_0 + re^{i\theta})$ (for fixed r.)

Cor 7.2 & 7.3 $f = u + iv$ holo. in $D_R(z_0)$,

Then $$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta, \quad \forall\ 0 < r < R.$$

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + re^{i\theta})\, d\theta, \quad \forall\ 0 < r < R$$

These are mean-value property for holomophic and harmonic function respectively.

(End of Review)

Ch4 The Fourier Transform

§1 The Class $\mathcal{F}$

Def: $\forall a>0$, let $S_a = \{z \in \mathbb{C} : |\mathrm{Im}(z)| < a\}$ (a horizontal strip)

Then

$$\mathcal{F}_a = \left\{ f : S_a \to \mathbb{C} : \begin{array}{l} f \text{ holo. on } S_a \text{ and } \exists A>0 \text{ s.t.} \\ |f(x+iy)| \le \dfrac{A}{1+x^2},\ \forall x \in \mathbb{R}\ \&\ |y|<a \end{array} \right\}$$

and $\mathcal{F} = \bigcup_{a>0} \mathcal{F}_a$

Remark: For a fixed y, with $|y|<a$, the condition that

$$\exists A>0 \text{ s.t. } |f(x+iy)| \le \frac{A}{1+x^2},\ \forall x \in \mathbb{R}$$

is usually referred as "moderate decay" on the horizontal line $\mathrm{Im}(z) = y$.

Hence, $f \in \mathcal{F}_a$ are moderate decay for each horizontal line $\mathrm{Im}(z) = y$, uniformly in $|y|<a$.

egs (i) Clearly $f(z) = e^{-\pi z^2} \in \mathcal{F}_a$, $\forall a>0$ (Ex!)

(ii) $\forall c>0$, the function

$$f(z)=\frac{1}{\pi}\frac{c}{c^2+z^2}\in\mathfrak{F}_a,\ \forall a\in(0,c).\ (\text{Ex!})$$

Clearly, $f(z)\notin\mathfrak{F}_a$ for $a\geq c$ as $z=\pm ci$ are poles.

Remarks: (1) For integer $n\geq 1$, $f\in\mathfrak{F}_a \Rightarrow f^{(n)}\in\mathfrak{F}_b$, $\forall\, 0<b<a$.

(Ex 2 of Ch 4 of Text)

(2) Many results in this chapter remain unchanged under the following weaker condition: $(\varepsilon>0)$

$$\exists A>0 \text{ s.t. } |f(x+iy)|\leq\frac{A}{1+|x|^{1+\varepsilon}}\quad \forall x\in\mathbb{R}\ \&\ |y|<a.$$

(Omitted)

§2 Action of the Fourier Transform on $\mathfrak{F}$

Def Let $f:\mathbb{R}\to\mathbb{C}$. The Fourier transform of f is

$$\hat{f}(\xi)=\int_{-\infty}^{\infty}f(x)e^{-2\pi ix\xi}dx,\quad \xi\in\mathbb{R}$$

For $f \in \mathfrak{F}_a$, we consider the Fourier transform of $f(x)$,

i.e. when $y=0$. Then we have

> **Thm 2.1** If $f \in \mathfrak{F}_a$, for some $a>0$, then $\exists B>0$ s.t.
>
> $$|\hat{f}(\xi)| \leq B e^{-2\pi b|\xi|}, \quad \forall\ 0 \leq b < a.$$

Pf: $|\hat{f}(\xi)| \leq \int_{-\infty}^{\infty} |f(x)|\,dx$ since $x, \xi \in \mathbb{R}$

$\leq \int_{-\infty}^{\infty} \frac{A}{1+x^2}\,dx = B$ which is bounded,

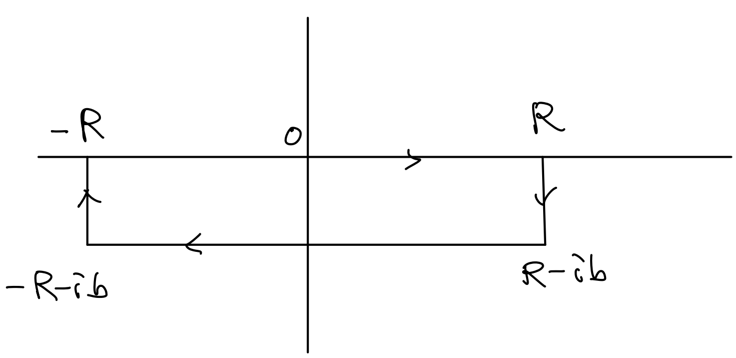
$\Rightarrow \quad |\hat{f}(\xi)| \leq B e^{-2\pi b|\xi|}$ holds for $b=0$.

For $0<b<a$:

If $\xi>0$, consider the contour integral of the holo. function

$g(z) = f(z) e^{-2\pi i z \xi}$ in S_a along the contour

which is the boundary of the rectangle $[-R,R]\times[-b,0]$ $(R>0)$



On the vertical edge $[-R-ib, -R]$,

using parametrization $z(t) = -R - it$, $t \in [0, b]$ (reverse direction)

$$\left| \int_{-R-ib}^{-R} g(z)\,dz \right| \leq \int_0^b \left| f(-R-it)\, e^{-2\pi i(-R-it)\xi} \right| dt$$

$$\leq \int_0^b \frac{A}{1+R^2} e^{-2\pi \xi t}\, dt \qquad \text{since } \xi > 0$$

$$= \frac{A}{1+R^2} \int_0^b e^{-2\pi \xi t}\, dt \longrightarrow 0 \quad \text{as } R \to +\infty.$$

Similarly $\left| \int_R^{R-ib} g(z)\,dz \right| \leq \frac{A}{1+R^2} \int_0^b e^{-2\pi\xi t}\, dt \longrightarrow 0$ as $R \to +\infty$.

Therefore, Cauchy theorem $\Rightarrow$

$$\left| \int_{-R}^{R} f(x) e^{-2\pi i x \xi}\, dx - \int_{-R}^{R} f(x-ib) e^{-2\pi i (x-ib)\xi}\, dx \right| \leq \frac{2A}{1+R^2} \int_0^b e^{-2\pi\xi t}\, dt.$$

Letting $R \to +\infty$, we have

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx$$

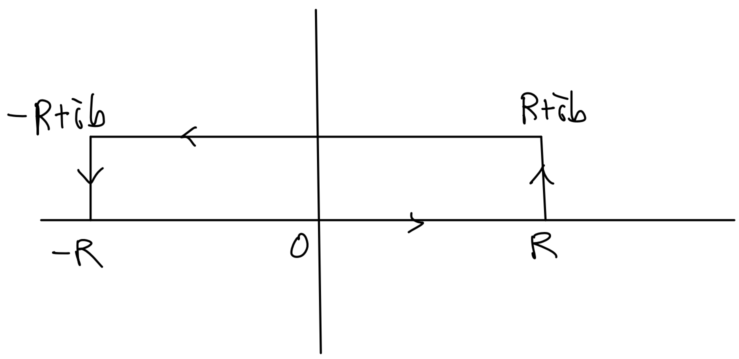
$$= \int_{-\infty}^{\infty} f(x-ib) e^{-2\pi i (x-ib)\xi}\, dx \quad \text{———} \quad (*),$$

$$\Rightarrow \quad |\hat{f}(\xi)| \leq \int_{-\infty}^{\infty} |f(x-ib)|\, e^{-2\pi b \xi}\, dx \qquad (\xi > 0)$$

$$\leq \left(\int_{-\infty}^{\infty} \frac{A}{1+x^2}\,dx\right) e^{-2\pi b\xi}$$

$$= Be^{-2\pi b|\xi|} \qquad (\xi>0)$$

For $\xi<0$, consider similarly the contour integral of $g(z)$ along:



$$\Rightarrow \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x+ib)\, e^{-2\pi i(x+ib)\xi}\,dx \quad \text{———} \quad (*)_2 \quad (\text{Ex!})$$

and hence the result.

※

<u>Remark</u>: Therefore, if $f \in \mathfrak{F} = \bigcup_{a>0} \mathfrak{F}_a$, then

$|\hat{f}(\xi)|$ decay exponentially as $|\xi| \to +\infty$,

in particular, it is <u>rapid decay at infinity</u> (i.e. decay faster than any $|\xi|^{-N}$, $\forall N>0$. More precisely $o\left(\frac{1}{|\xi|^N}\right)$, $\forall N>0$.)

Thm 2.2 (Fourier Inversion Formula)

If $f \in \mathcal{F}$, then

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi, \quad \forall x \in \mathbb{R}$$

The proof needs a lemma:

Lemma 2.3 If $A > 0$ & $B \in \mathbb{R}$, then

$$\int_0^{\infty} e^{-(A+iB)\xi} d\xi = \frac{1}{A+iB}$$

Pf:

Note $A>0, B \in \mathbb{R} \Rightarrow |e^{-(A+iB)\xi}| = e^{-A\xi}$, for $\xi \in (0,\infty)$.

Hence the improper integral converges.

$$\therefore \int_0^{\infty} e^{-(A+iB)\xi} d\xi = \lim_{R \to +\infty} \int_0^R e^{-(A+iB)\xi} d\xi$$

$$= \lim_{R \to +\infty} \left[\frac{e^{-(A+iB)\xi}}{-(A+iB)} \right]_0^R = \frac{1}{A+iB}$$

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<u>Pf of Thm 2.2 (Fourier Inversion Formula)</u>

Note that $f \in \mathfrak{F} \Rightarrow f \in \mathfrak{F}_a$ for some $a > 0$.

Then by equations $(*)_1$ & $(*)_2$ in the proof of Thm 2.1,

(eqt (1) in the text)

If $\xi > 0$, $\quad \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x-ib)\, e^{-2\pi i (x-ib)\xi}\, dx, \quad \forall\, 0 < b < a.$

If $\xi < 0$, $\quad \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x+ib)\, e^{-2\pi i (x+ib)\xi}\, dx, \quad \forall\, 0 < b < a$

As sign of ξ is important in the proof, we write

$$\int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x\xi}\, d\xi = \int_{-\infty}^{0} \hat{f}(\xi) e^{2\pi i x \xi}\, d\xi + \int_{0}^{\infty} \hat{f}(\xi)\, e^{2\pi i x \xi}\, d\xi$$

and work on the integrals individually:

$$\int_0^{\infty} \hat{f}(\xi) e^{2\pi i x\xi}\, d\xi = \int_0^{\infty} \left(\int_{-\infty}^{\infty} f(u-ib)\, e^{-2\pi i (u-ib)\xi}\, du \right) e^{2\pi i x \xi}\, d\xi$$

Since $|f(u-ib)| \leq \frac{A}{1+u^2}$, (for some $A > 0$)

the (iterated) integrals are absolute convergence.

Hence Fubini $\Rightarrow$

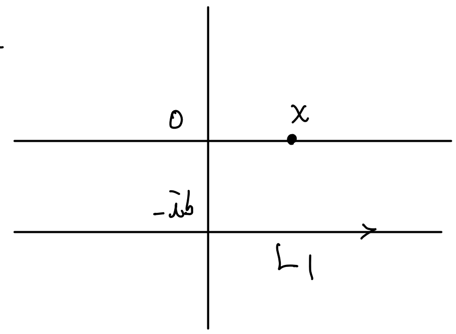
$$\int_0^\infty \hat{f}(\xi) e^{2\pi i x \xi} d\xi = \int_{-\infty}^{\infty} f(u-ib) \int_0^\infty e^{-2\pi i (u-ib)\xi} e^{2\pi i x \xi} d\xi \, du$$

$$= \int_{-\infty}^{\infty} f(u-ib) \left(\int_0^\infty e^{-2\pi i (u-x-ib)\xi} d\xi \right) du$$

(Lemma 2.3) $$= \int_{-\infty}^{\infty} f(u-ib) \frac{1}{2\pi b + 2\pi i (u-x)} du$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(u-ib)}{(u-ib)-x} du$$

$$= \frac{1}{2\pi i} \int_{L_1} \frac{f(\zeta)}{\zeta - x} d\zeta$$



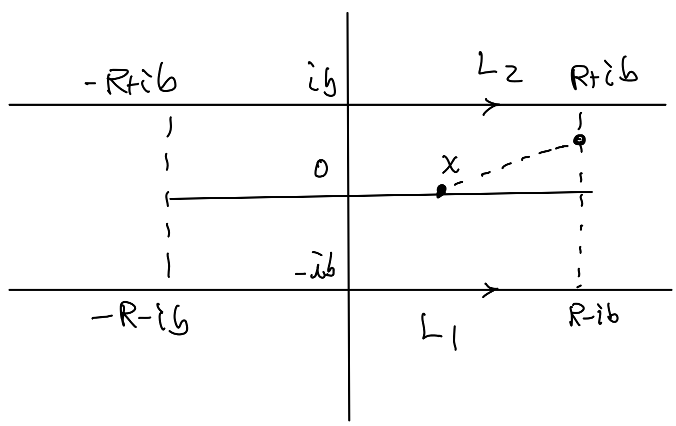
The contour integral of $\frac{f(z)}{z-x}$ (x fixed)

along the horizontal line $y=-b$ (from left to right)

Similarly for $\xi < 0$,

$$\int_{-\infty}^0 \hat{f}(\xi) e^{2\pi i x \xi} d\xi$$

$$= -\frac{1}{2\pi i} \int_{L_2} \frac{f(\zeta)}{\zeta - x} d\zeta$$



where $L_2 : \{y=b\}$ from left to right.

Note that

$$\left|\int_{R-ib}^{R+ib} \frac{f(\zeta)}{\zeta - x}\, d\zeta\right| \leq 2b\,\frac{A}{1+R^2}\cdot\frac{1}{R-x} \qquad \text{for } R > x.$$

$$\longrightarrow 0 \quad \text{as } R \to +\infty$$

Similarly,

$$\left|\int_{-R-ib}^{-R+ib} \frac{f(\zeta)}{\zeta - x}\, d\zeta\right| \to 0 \quad \text{as } R \to +\infty.$$

Cauchy integral formula, by letting $R \to +\infty$,

$$f(x) = \frac{1}{2\pi i}\int_{L_1} \frac{f(\zeta)}{\zeta - x}\, d\zeta - \frac{1}{2\pi i}\int_{L_2} \frac{f(\zeta)}{\zeta - x}\, d\zeta$$

$$= \int_0^{\infty} \hat{f}(\xi) e^{2\pi i x\xi}\, d\xi + \int_{-\infty}^{0} \hat{f}(\xi) e^{2\pi i x\xi}\, d\xi$$

$$= \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i x\xi}\, d\xi.$$

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