Review (Ch1–3 of the Textbook)

Ch1 Preliminaries to Cpx Analysis

§1 Cpx numbers & Cpx plane (Self reading)

Recall notations:

- open disc of radius r centered at z_0 : $D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$
- closed disc of radius r centered at z_0 : $\overline{D}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$
- boundary of $D_r(z_0)$ (or $\overline{D}_r(z_0)$) : $C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$
- unit disc : $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$
- diameter of a set $\Omega \subset \mathbb{C}$: $\operatorname{diam}(\Omega) = \sup_{z, w \in \Omega} |z - w|$
- region = open connected set in $\mathbb{C}$

§2 Functions of the Cpx plane

2.1 Self reading

2.2 Holomorphic functions

- Ω open set in $\mathbb{C}$,
- f cpx-valued function on Ω.

Def: f is holomorphic at the point $z_0 \in \Omega$ if

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} \text{ exists.}$$

$(h \in \mathbb{C},\ h \neq 0 \text{ s.t. } z_0 + h \in \Omega)$

And if it exists, it is called the derivative of f at z_0

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$$

- f is said to be holomorphic on Ω if f is holomorphic at z_0, $\forall z_0 \in \Omega$.

- If C is a closed set in $\mathbb{C}$, then f is holomorphic on C if $\exists$ open set Ω s.t. $C \subset \Omega$ and f is holomorphic on Ω.

- f is called entire if f is holomorphic on $\mathbb{C}$

- Cauchy-Riemann equations

If $f = u + iv$ holomorphic on Ω (open), (u, v $\mathbb{R}$-valued)

then
$$\boxed{\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad \text{on } \Omega}$$

- cpx differential operators $\frac{\partial}{\partial z}$ & $\frac{\partial}{\partial \bar{z}}$:

$$\boxed{\begin{aligned} \frac{\partial}{\partial z} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \\ \frac{\partial}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \end{aligned}}$$

- Then $\boxed{\text{Cauchy-Riemann} \Leftrightarrow \frac{\partial f}{\partial \bar{z}} = 0}$.

<u>Prop 2.3</u> $f = u + iv$ holomorphic at z_0, then

$$\begin{cases} \frac{\partial f}{\partial \bar{z}}(z_0) = 0 \\ \frac{\partial f}{\partial z}(z_0) = f'(z_0) = 2\frac{\partial u}{\partial z}(z_0) \end{cases}$$

Also $F : \Omega \to \mathbb{R}^2 : (x, y) \mapsto (u(x,y), v(x,y))$ is <u>differentiable</u>

(as $\Omega \to \mathbb{R}^2$ mapping)

and $\det J_F(x_0, y_0) = |f'(z_0)|^2$,

where J_F is the <u>Jacobian matrix</u> of F

<u>Thm 2.4</u> $f = u + iv$ defined on an <u>open</u> $\Omega \subset \mathbb{C}$,

(u, v are <u>real-valued</u> functions on Ω)

If <u>$u, v \in C^1(\Omega)$</u> and satisfy <u>Cauchy-Riemann</u> eqt.

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases} \quad \text{on } \Omega.$$

then f is <u>holomorphic on Ω</u> & $f' = \frac{\partial f}{\partial z}$.

2.3 Power series $\sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$

- absolute convergence (at z) if the real-valued series

$$\sum_{n=0}^{\infty} |a_n| |z|^n \text{ converges}$$

Thm 2.5 Given $\sum_{n=0}^{\infty} a_n z^n$, define

$$R = \frac{1}{\limsup |a_n|^{\frac{1}{n}}} \quad (\in [0, \infty])$$

then (i) If $|z| < R$, $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely

(ii) If $|z| > R$, $\sum_{n=0}^{\infty} a_n z^n$ diverges

Remarks :
- no conclusion on $\{|z| = R\}$
- R is called the radius of convergence
- $\{|z| < R\}$ the disc of convergence

Thm 2.6

$f(z) = \sum_{n=0}^{\infty} a_n z^n$ holomorphic on the disc of convergence (provided $R > 0$)

and

$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$ with the same radius of convergence.

Cor 2.7. $\sum_{n=0}^{\infty} a_n z^n$ infinitely (cpx) differentiable & higher derivatives can be calculated by termwise differentiation (in its disc of convergence)

Def $f: \Omega \xrightarrow{} \mathbb{C}$ (open) is (cpx) analytic at $z_0 \in \Omega$

if $\exists$ $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ with positive radius of convergence

such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{in a nbd. of } z_0.$$

Clearly by Thm 2.6, $\boxed{\text{(cpx) analytic} \Rightarrow \text{holomorphic}}$

§3 Integration along curves : Self reading.

$$\int_\gamma f(z)\,dz$$

Useful notation : $\begin{cases} dz = dx + i\,dy \\ d\bar{z} = dx - i\,dy \end{cases}$

Then • $\int_\gamma f\,dz = \int_\gamma (u+iv)(dx+i\,dy)$

$$= \int_\gamma (u\,dx - v\,dy) + i \int_\gamma (v\,dx + u\,dy)$$

- $df = du + i\,dv$

$$= f_x\,dx + f_y\,dy$$

$$= \frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial \bar{z}}\,d\bar{z}$$

$\left(\therefore\ f \text{ holo.} \Rightarrow df = f'\,dz \right)$

Ch2 Cauchy's Theorem & Its applications

§1 Goursat's Theorem

Thm 1.1 & Cor 1.2

If
- Ω open in $\mathbb{C}$,
- f holomorphic on Ω, (note the different in terminology in the textbook)
- γ = boundary of a triangle T or rectangle R

  s.t. $\gamma \cup T$ or $\gamma \cup R \subset \Omega$,

then $\int_\gamma f(z)\,dz = 0$

Remark: The main point in Goursat's Thm is that there is no need to assume f' is continuous. Cauchy's first observation used Green's Thm which need to assume u_x, u_y, v_x & v_y are continuous.

§2 Local existence of primitive & Cauchy's Theorem in a disc

(and Appendix B: Simply Connectivity and Jordan Curve Theorem)

Notation: For a simple closed piecewise smooth curve γ,

$\text{int}(\gamma)$ = bounded component of $\mathbb{C} \setminus \gamma$

(i.e. the interior of the Jordan curve of γ,

not the interior of γ as a topological point set)

> **Thm 2.9** (on page 361 of the text book)
>
> If
> - $f : \Omega \to \mathbb{C}$ is holo., Ω open,
> - γ = simple closed piecewise smooth curve s.t.
> - $\gamma \cup \text{int}(\gamma) \subset \Omega$
>
> Then $\int_\gamma f\, dz = 0$.

§3 Evaluation of some integrals (self reading)

§4 Cauchy's Integral Formula

Thm 4.1 & Cor 4.2

If
- f is holo. on Ω (open)
- C positive oriented simple closed piecewise smooth curve s.t.
- $C \cup \mathrm{int}(C) \subset \Omega$

then $\forall\, z \in \mathrm{int}(C)$ & $n = 0, 1, 2, \cdots$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta .$$

Consequences
- Cor 4.3 Cauchy inequalities
- Thm 4.4 Holomaphic $\Rightarrow$ analytic & Taylor's fumula
- Cor 4.5 Liouville's Theorem
- Cor 4.6 Foundamental Theorem of Algebra
- Cor 4.7 Factorization of Polynomial
- Thm 4.8 & Cor 4.9 uniqueness of holomaphic function

Self reading.

§5 Further applications

5.1 Morera's Thm (converse of Cauchy's Thm)

Thm 5.1 • f cts. on Ω & (note the diff. in terminology in the textbook)

• $\int_{\partial T} f = 0$ $\forall$ triangle T with $T \cup \partial T \subset \Omega$,

then f is holomorphic on Ω.

5.2 Sequence of Holomorphic Functions

Thm 5.2 & Thm 5.3

If • f_n holo. on Ω,

• $f_n \to f$ uniformly on cpt. subsets

Then f holo on Ω and

$f_n' \to f'$ uniformly on cpt. subsets.

5.3 Holomorphic functions defined in terms of integrals

Thm 5.4 • Ω open in $\mathbb{C}$,

• $F(z,s): \Omega \times [a,b] \to \mathbb{C}$.

Suppose (1) For each $s \in [a,b]$, $F(z,s)$ is holo. in z.

(2) $F \in C(\Omega \times [a,b])$.

Then

$$f(z) = \int_a^b F(z,s)\,ds$$

is a holomorphic function on Ω.

(The proof is not covered in MATH2230)

Pf: It is clear that one may assume $[a,b] = [0,1]$.

Since Ω may be unbounded, we work on an arbitrary disc $D \subset \overline{D} \subset \Omega$.

For $n \geq 1$, consider Riemann sum

$$f_n(z) = \frac{1}{n} \sum_{k=1}^{n} F(z, \tfrac{k}{n})$$

Then, (1) $\Rightarrow$ $f_n(z)$ is holo. $\forall\, n \geq 1$.

By (2), $F \in C(\Omega \times [0,1])$

$\Rightarrow$ $F(z,s)$ is uniformly continuous on $\overline{D} \times [0,1]$.

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists\, \delta > 0$ s.t. $\forall z \in \overline{D}$

$$|F(z,s_1) - F(z,s_2)| < \varepsilon, \quad \forall\, |s_1 - s_2| < \delta$$

(Since $\operatorname{dist}((z,s_1),(z,s_2)) = |s_1 - s_2| < \delta$)

$$\Rightarrow \sup_{z \in D} |F(z,s_1) - F(z,s_2)| < \varepsilon, \quad \forall\, |s_1 - s_2| < \delta.$$

Therefore, if $n > \frac{1}{\delta}$, then $\forall z \in D \subset \overline{D}$

$$|f_n(z) - f(z)| = \left| \frac{1}{n} \sum_{k=1}^{n} F(z, \tfrac{k}{n}) - \int_0^1 F(z,s)\,ds \right|$$

$$= \left| \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} F(z, \tfrac{k}{n})\,ds - \int_0^1 F(z,s)\,ds \right|$$

$$= \left| \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left(F(z, \tfrac{k}{n}) - F(z,s) \right) ds \right|$$

$$\leq \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} \left| F(z, \tfrac{k}{n}) - F(z,s) \right| ds$$

$$< \varepsilon \sum_{k=1}^{n} \int_{\frac{k-1}{n}}^{\frac{k}{n}} ds = \varepsilon$$

$\therefore$ f is the uniform limit of f_n on D

By Thm 5.2 & 5.3, f is holomorphic on D.

Since $D \subset \overline{D} \subset \Omega$ is arbitrary, f is holomorphic on Ω. ※