1.1 Analytic Continuation

Lemma 1.2 If $Re(s)>0$, then

$$\Gamma(s+1) = s\Gamma(s) \,. \qquad \text{(2)}$$

Hence $\quad \Gamma(n+1) = n!$ for $n=0,1,2,3,\cdots$

Pf: For $Re(s)>0$,

$$\Gamma(s+1) = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\frac{1}{\varepsilon}} e^{-t} t^s \, dt \,.$$

$$= \lim_{\varepsilon \to 0} \left\{ \left[-e^{-t} t^s \right]_{\varepsilon}^{\frac{1}{\varepsilon}} + \int_{\varepsilon}^{\frac{1}{\varepsilon}} e^{-t} \cdot s t^{s-1} \, dt \right\}$$

$$= \lim_{\varepsilon \to 0} \left[\left(e^{-\varepsilon} \varepsilon^s - e^{-\frac{1}{\varepsilon}} \left(\frac{1}{\varepsilon}\right)^s \right) + s \int_{\varepsilon}^{\frac{1}{\varepsilon}} e^{-t} t^{s-1} dt \right]$$

$$= s\Gamma(s),$$

since $Re(s) > 0 \Rightarrow$

$$\begin{cases} |e^{-\varepsilon} \varepsilon^s| = e^{-\varepsilon} \varepsilon^{Re(s)} \to 0 \\ \left|e^{-\frac{1}{\varepsilon}} \left(\frac{1}{\varepsilon}\right)^s\right| = e^{-\frac{1}{\varepsilon}} \left(\frac{1}{\varepsilon}\right)^{Re(s)} \to 0 \end{cases} \quad \text{as } \varepsilon \to 0 \,.$$

This proves formula (2).

By formula (2), if $n \geq 1$, then

$$\Gamma(n+1) = n\Gamma(n) = \cdots = n(n-1)\cdots 1 \cdot \Gamma(1)$$

$$= n! \, \Gamma(1) \,.$$

And $\Gamma(1) = \int_0^\infty e^{-t} t^{1-1} dt = \int_0^\infty e^{-t} dt = 1$

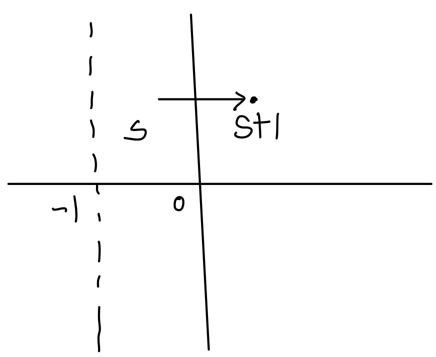
$\therefore \Gamma(n+1) = n! \quad (n \geq 1)$

For $n=0$, $\quad \Gamma(0+1) = 1 = 0!$ by definition. #

Thm 1.3 The gamma function $\Gamma(s)$ defined for $\mathrm{Re}(s) > 0$ has an analytic continuation to a meromorphic function on $\mathbb{C}$ whose only singularities are simple poles at $s = 0, -1, -2, \cdots$ with residue

$$\mathrm{res}_{s=-n} \Gamma(s) = \frac{(-1)^n}{n!}$$

Remark: Since $\mathbb{C} \setminus \{0, -1, -2, \cdots\}$ is connected, the analytic continuation of $\Gamma(s)$ is unique (by Thm 4.8 & Cor 4.9 of Ch 2). Therefore, it is convenient to denoted this analytic continuation by $\Gamma(s)$ again. So after proving this Theorem, the gamma function $\Gamma(s)$ is a meromorphic function on $\mathbb{C}$.



Pf: For $Re(s) > -1$, define

$$F_1(s) = \frac{\Gamma(s+1)}{s}$$

Since $\Gamma(s)$ holo in $Re(s) > 0$,

$\Gamma(s+1)$ holo in $Re(s) > -1$,

and hence $F_1(s) = \frac{\Gamma(s+1)}{s}$ is meromophic in $Re(s) > -1$

with a simple pole at $s=0$ with

$$\operatorname{res}_{s=0} F_1(s) = \Gamma(0+1) = 1.$$

Note that Lemma 1.2 $\Rightarrow$ $F_1(s) = \frac{\Gamma(s+1)}{s} = \Gamma(s)$ for $Re(s) > 0$,

$F_1(s)$ is an analytic continuation of $\Gamma(s)$ to a mero. function

on $\{ s \in \mathbb{C} : Re(s) > -1 \}$.

Same argument works for $Re(s) > -m$ by defining

$$F_m(s) = \frac{\Gamma(s+m)}{(s+m-1)(s+m-2)\cdots s}.$$

Clearly $F_m(s)$ is meromorphic in $Re(s) > -m$ ($\Rightarrow Re(s+m) > 0$)

with simple poles at $s = 0, -1, \cdots, -(m-1)$,

and for $n = 0, 1, \cdots, m-1$

$$\operatorname{res}_{s=-n} F_m(s) = \frac{\Gamma(-n+m)}{(-n+m-1)(-n+m-2)\cdots(1)(-1)(-2)\cdots(-n)}$$

the term corresponding to the pole

$$= \frac{\Gamma(m-n)}{(m-n-1)!\,(-1)^n n!}$$

$$= \frac{(-1)^n}{n!} \quad \text{by Lemma 1.2.}$$

And for $Re(s)>0$,

$$F_m(s) = \frac{\Gamma(s+m)}{(s+m-1)(s+m-2)\cdots s} = \frac{(s+m-1)\Gamma(s+m-1)}{(s+m-1)(s+m-2)\cdots s} \quad \text{(by Lemma 1.2)}$$

$$= F_{m-1}(s) \cdots = F_1(s) = \Gamma(s)$$

$\therefore$ $F_m(s)$ is an analytic continuation of $\Gamma(s)$ to $\{Re(s)>-m\}$.

Then uniqueness of theorem $\Rightarrow$ if $m>n$, then

$$F_m(s) = F_n(s) \quad \text{for } Re(s)>-n.$$

Therefore, one can define meromorphic function $F(s)$ on $\mathbb{C}$

by

$$F(s) \overset{\text{def}}{=\!=} F_m(s) \quad \text{if } Re(s)>-m.$$

Clearly, this gives the required analytic continuation. #

Remarks: (1) Clearly $\lim_{s\to 0} s\Gamma(s) = \Gamma(1) = 1$

(2) $\Gamma(s+1) = s\Gamma(s)$ holds for $s \in \mathbb{C} \setminus \{-1,-2,\cdots\}$.

Pf: LHS holo. except $s+1 = 0,-1,-2,\cdots$

RHS holo. except $s = -1, -2, \cdots$

(since $s=0$ is a simple pole hence it is removable for RHS.)

And on $\{Re(s) > 0\}$, LHS = RHS. Therefore uniqueness thm $\Rightarrow$ LHS $\equiv$ RHS on $\mathbb{C} \setminus \{-1, -2, \cdots\}$.

(3) $\operatorname{res}_{s=-n} \Gamma(s+1) = -n \operatorname{res}_{s=-n} \Gamma(s)$ $(n = 1, 2, 3, \cdots)$

Pf: near $s = -(n-1)$,

$$\Gamma(s) = \frac{\frac{(-1)^{n-1}}{(n-1)!}}{s+(n-1)} + \text{holo}(s)$$

$$\Rightarrow \quad \Gamma(s+1) = \frac{\frac{(-1)^{n-1}}{(n-1)!}}{s+n} + \text{holo}(s+1)$$

$$\therefore \quad \operatorname{res}_{s=-n} \Gamma(s+1) = \frac{(-1)^{n-1}}{(n-1)!} = -n \cdot \operatorname{res}_{s=-n} \Gamma(s).$$

#

<u>Alternating Proof of Thm 1.3</u>: $\forall s \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}$,

$$\boxed{\Gamma(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{n+s} + \int_1^{\infty} e^{-t} t^{s-1} dt \quad \text{——— (3)}}$$

Pf: We 1st show (3) for $Re(s) > 0$.

By Prop 1.1 and formula (1), for $Re(s) > 0$,

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\,dt$$

$$= \int_0^1 e^{-t} t^{s-1}\,dt + \int_1^\infty e^{-t} t^{s-1}\,dt$$

For $t \in (0,1)$, $\quad e^{-t} = \sum_{n=0}^\infty \frac{(-t)^n}{n!} = \sum_{n=0}^\infty \frac{(-1)^n}{n!} t^n$

By absolute convergence of the improper integral and uniform convergence of the series, we have

$$\int_0^1 e^{-t} t^{s-1}\,dt = \int_0^1 \left(\sum_{n=0}^\infty \frac{(-1)^n}{n!} t^n \right) t^{s-1}\,dt$$

$$= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_0^1 t^{n+s-1}\,dt$$

$$= \sum_{n=0}^\infty \frac{(-1)^n}{n!} \cdot \frac{1}{n+s}$$

$$\therefore \quad \Gamma(s) = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \cdot \frac{1}{n+s} + \int_1^\infty e^{-t} t^{s-1}\,dt, \quad \forall \operatorname{Re} s > 0.$$

Now clearly $\int_1^\infty e^{-t} t^{s-1}\,dt$ is an <u>entire</u> function because of the exponential decay (Ex!).

For the series, consider any $R > 0$ and any $N > 2R$

$$\sum_{n=0}^\infty \frac{(-1)^n}{n!} \cdot \frac{1}{n+s} = \sum_{n=0}^N \frac{(-1)^n}{n!} \cdot \frac{1}{n+s} + \sum_{n=N+1}^\infty \frac{(-1)^n}{n!} \cdot \frac{1}{n+s}$$

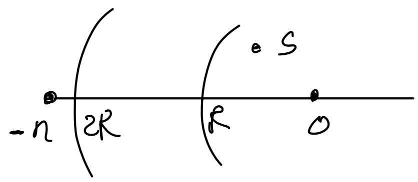
$\sum_{n=0}^{N} \frac{(-1)^n}{n!} \cdot \frac{1}{n+s}$ is a meromorphic function in $\{|s|<R\}$ with poles at

$$k \in \{0,-1,-2,\cdots,-N\} \cap \{k : |k|<R\}$$

$\sum_{n=N+1}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{n+s}$ has general term

$$\left| \frac{(-1)^n}{n!} \cdot \frac{1}{n+s} \right| \leq \frac{1}{n!} \cdot \frac{1}{R}$$

since $n > N > 2R$ and $|s|<R$



$\Rightarrow \quad |n+s| > R$

$\therefore \quad \sum_{n=N+1}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{n+s}$ uniformly converges to a holomorphic function in $\{|s|<R\}$.

Since $R>0$ is arbitrary,

$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{n+s} + \int_1^{\infty} e^{-t} t^{s-1} dt$ defines a meromorphic function with simple poles at $s = \{0,-1,-2,\cdots\}$

with $$\operatorname{res}_{s=-n} = \frac{(-1)^n}{n!}.$$

Since it $= \Gamma(s)$ for $\operatorname{Re} s > 0$, we've proved (3), $\forall s \in \mathbb{C} \setminus \{0,-1,\cdots\}$.

#

1.2 Further Properties of $\Gamma(s)$

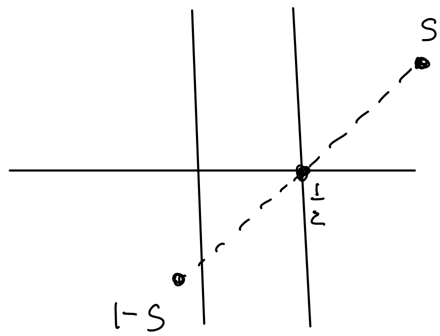
Thm 1.4 $\forall s \in \mathbb{C} \setminus \mathbb{Z}$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} \qquad \text{———} \quad (4)$$

Remark:

(1) $s \mapsto 1-s$

is the reflection across $\frac{1}{2}$



$\mathrm{Re}\, s > \frac{1}{2}$

(2) Thm 1.4 $\Rightarrow$ $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. (Since $\Gamma(s) > 0$, $\forall s > 0$)

Pf: Note that $\Gamma(s)$ mero. with simple poles at $s = 0, -1, -2, \cdots$

$\Rightarrow$ $\Gamma(1-s)$ mero. with simple poles at $s = 1, 2, 3, \cdots$

$\therefore$ $\Gamma(s)\Gamma(1-s)$ is mero. with simple poles at $s \in \mathbb{Z}$.

Clearly, $\dfrac{\pi}{\sin \pi z}$ is also mero. with simple poles at $s \in \mathbb{Z}$.

Therefore, by connectedness of $\mathbb{C} \setminus \mathbb{Z}$, it suffices to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi z}$$

on a subset of $\mathbb{C} \setminus \mathbb{Z}$ with accumulation points.

Note that for $0 < s < 1$ (a subset with accumulation points)

$$\Gamma(1-s) = \int_0^\infty e^{-u} u^{(1-s)-1} du$$ (changed the notation of the dummy variable)

$$= \int_0^\infty e^{-u} u^{-s} du$$

$$= \int_0^\infty e^{-tv} (tv)^{-s} t\, dv \qquad \forall t>0$$

$$\Rightarrow \quad \Gamma(1-s)\Gamma(s) = \Gamma(1-s) \int_0^\infty e^{-t} t^{s-1} dt$$

$$= \int_0^\infty e^{-t} t^{s-1} \Gamma(1-s)\, dt$$

$$= \int_0^\infty e^{-t} t^{s-1} \left(\int_0^\infty e^{-tv} (tv)^{-s} t\, dv \right) dt$$

$$= \int_0^\infty \int_0^\infty e^{-t(1+v)} v^{-s} dv\, dt$$

$$= \int_0^\infty \left(\int_0^\infty e^{-t(1+v)} dt \right) v^{-s} dv$$

(exponential decay at $t, v \to \infty$ and integrability of $\int_0^1 v^{-s} ds$ $(0<s<1)$
$\Rightarrow$ integrals are absolutely integrable and hence
Fubini theorem applies.)

$$\therefore \quad \Gamma(1-s)\Gamma(s) = \int_0^\infty \frac{1}{1+v} v^{-s} dv$$

$$= \int_{-\infty}^\infty \frac{e^{(1-s)x}}{1+e^x} dx$$

(using $0<s<1$ $\Leftrightarrow$ $0<1-s<1$)

$$= \frac{\pi}{\sin \pi(1-s)}$$ (by Eg 2 of §2.1 of Ch3 of the Text page 79)

$$= \frac{\pi}{\sin \pi s}$$

Therefore, uniqueness thm $\Rightarrow$ $\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin \pi s}$, $\forall s \in \mathbb{C} \setminus \mathbb{Z}$.

#

Thm 1.6 (i) $1/\Gamma(s)$ is an entire function of s with simple zeros at $s = 0, -1, -2, \cdots$ &

$1/\Gamma(s) \neq 0$ for $s \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}$.

(ii) $\left|\frac{1}{\Gamma(s)}\right| \leq C_1 e^{C_2 |s| \log |s|}$, for some constants $C_1, C_2 > 0$.

$\Rightarrow$ $1/\Gamma(s)$ is of order 1:

$\forall \varepsilon > 0$, $\exists$ $C = C(\varepsilon) > 0$ s.t. $\left|\frac{1}{\Gamma(s)}\right| \leq C(\varepsilon) e^{C_2 |s|^{1+\varepsilon}}$.

Pf: By Thm 1.4, $\frac{1}{\Gamma(s)} = \Gamma(1-s) \frac{\sin \pi s}{\pi}$

Note that $\Gamma(1-s)$ has simple poles at $s = 1, 2, 3, \cdots$ $(1-s = 0, -1, -2, \cdots)$

and $\sin \pi s$ has simple zeros at $s = 1, 2, 3, \cdots$

So $s = 1, 2, 3, \cdots$ are removable singularities for $1/\Gamma(s)$.

Together with the fact the Γ has no other singularity,

$\frac{1}{\Gamma(s)}$ is entire,

and vanishing only at $s = 0, -1, -2, \cdots$ (the simple poles of $\Gamma(s)$).

This proves (i).

To prove (ii), we are going to use formula (3) (alternative proof of Thm 1.3)

$$\Gamma(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{n+s} + \int_1^{\infty} e^{-t} t^{s-1} dt$$

which implies

$$\Gamma(1-s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{n+1-s} + \int_1^{\infty} e^{-t} t^{-s} dt$$

and hence

$$\frac{1}{\Gamma(s)} = \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{n+1-s} \right) \frac{\sin \pi s}{\pi} + \left(\int_1^{\infty} e^{-t} t^{-s} dt \right) \frac{\sin \pi s}{\pi}$$

For simplicity, let $\sigma = \mathrm{Re}\, s$.

Then $$\left| \int_1^{\infty} e^{-t} t^{-s} dt \right| \leq \int_1^{\infty} e^{-t} t^{-\sigma} dt \leq \int_1^{\infty} e^{-t} t^{|\sigma|} dt$$

Choose $n \in \mathbb{N}$ s.t. $|\sigma| \leq n \leq |\sigma| + 1$.

Then $$\int_1^{\infty} e^{-t} t^{|\sigma|} dt \leq \int_1^{\infty} e^{-t} t^n dt \leq \int_0^{\infty} e^{-t} t^n dt$$

$$= \Gamma(n+1) = n! \qquad \text{(Lemma 1.2)}$$

$$\leq n^n = e^{n \log n}$$

$$\leq e^{(|\sigma|+1) \log(|\sigma|+1)} \leq e^{(|s|+1) \log(|s|+1)}$$

We also have $|\sin \pi s| \leq e^{\pi |s|}$ (eg 1 of §2 of Ch5).

$\therefore$ The 2nd term of $1/\Gamma(s)$ has bound

$$C e^{(|s|+1) \log(|s|+1)} \cdot e^{\pi |s|} \leq C_1 e^{C_2 |s| \log |s|}$$

for some constants $C_1, C_2 > 0$. (Ex!)

For the 1^{st} term $\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{n+1-s}\right) \frac{\sin \pi s}{\pi}$:

Case 1 $(|\mathrm{Im}(s)| > 1)$

Then $|n+1-s| \geq |\mathrm{Im}\, s| > 1$,

$$\left| \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{n+1-s} \right| \leq \sum_{n=0}^{\infty} \frac{1}{n!} = e$$

$\therefore$ The 1^{st} term $\leq C e^{\pi |s|}$ for some $C > 0$.

Case 2 $|\mathrm{Im}(s)| \leq 1$.

Choose $k \in \mathbb{Z}$ s.t. $k - \frac{1}{2} \leq \mathrm{Re}(s) < k + \frac{1}{2}$.

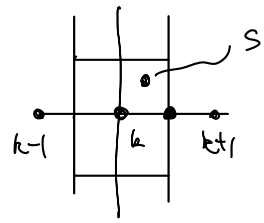
If $k \geq 1$,

$$\left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{n+1-s}\right) \frac{\sin \pi s}{\pi} = \frac{(-1)^{k-1}}{(k-1)!} \cdot \frac{1}{k-s} \frac{\sin \pi s}{\pi} \qquad (k = n+1)$$

$$+ \sum_{n \neq k-1} \frac{(-1)^n}{n!} \frac{1}{n+1-s} \cdot \frac{\sin \pi s}{\pi}$$

Note $\left| \frac{(-1)^{k-1}}{(k-1)!} \frac{1}{k-s} \frac{\sin \pi s}{\pi} \right| \leq C$ since $\frac{\sin \pi s}{k-s}$ has removable singularity at $s = k$ $\left(k - \frac{1}{2} \leq \mathrm{Re}\, s \leq k + \frac{1}{2} \;\&\; |\mathrm{Im}(s)| \leq 1 \right)$

(C is independent of k or s since $|\sin \pi s|$ is periodic) and $\left| \sum_{n \neq k-1} \frac{(-1)^n}{n!} \frac{1}{n+1-s} \cdot \frac{\sin \pi s}{\pi} \right|$



$|n+1-s| \geq \frac{1}{2}, \ \forall\, n \neq k-1$

$$\leq 2 \sum_{n=0}^{\infty} \frac{1}{n!} \frac{|\sin \pi s|}{\pi}$$

$$\leq C$$

since $|\operatorname{Im} s| \leq 1$ (& periodic of $\sin \pi s$).

If $k \leq 0$, then $k - \frac{1}{2} \leq \operatorname{Re} s \leq k + \frac{1}{2} \Rightarrow \operatorname{Re} s \leq \frac{1}{2}$.

$$\Rightarrow \quad |n+1-s| \geq \frac{1}{2}, \quad \forall\, n = 0, 1, 2, \cdots$$

$$\Rightarrow \left| \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{n+1-s} \cdot \frac{\sin \pi s}{\pi} \right| \leq 2 \sum_{n=0}^{\infty} \frac{1}{n!} \frac{|\sin \pi s|}{\pi} \leq C$$

since $|\operatorname{Im} s| \leq 1$ (& periodic of $\sin \pi s$).

All together

$$1/\Gamma(s) \leq C e^{\pi |s|} + C_1 e^{C_2 |s| \log |s|}$$

$$\Rightarrow \quad \frac{1}{\Gamma(s)} \leq C_1 e^{C_2 |s| \log |s|}$$ (maybe for new $C_1, C_2 > 0$)

This proves the 1st statement of (ii).

The 2nd statement of (ii) follows from $|s| \log |s| \leq C |s|^{1+\varepsilon}, \ \forall \varepsilon > 0$,

and $\sum \frac{1}{n^\sigma}$ converges $\Leftrightarrow \sigma > 1$,

and Thm 2.1 of §2 of Ch5. (Ex!)

#

Thm 1.7 $\forall s \in \mathbb{C}$

$$\frac{1}{\Gamma(s)} = e^{\gamma s}\, s \prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}},$$

where $\gamma = \lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n} - \log N\right)$ is the Euler's const.

Pf: By Hadamard factorization theorem (Thm 5.1 in §5 of Ch5) & Thm 1.6,

$$\frac{1}{\Gamma(s)} = e^{As+B}\, s \prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}$$

By remark (1) after the proof of Thm 1.3, $\lim_{s\to 0} s\Gamma(s) = 1$.

$$\Rightarrow \quad 1 = e^{B}, \text{ ie } B = 0 \ (\text{or } B = 2\pi i k,\ k\in\mathbb{Z})$$

$$\therefore \quad \frac{1}{\Gamma(s)} = e^{As}\, s \prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}$$

Putting $s=1$, we have

$$1 = \frac{1}{\Gamma(1)} = e^{A}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)e^{-\frac{1}{n}}$$

$$\Rightarrow \quad e^{-A} = \lim_{N\to\infty}\prod_{n=1}^{N}\left(1+\frac{1}{n}\right)e^{-\frac{1}{n}} = \lim_{N\to\infty}\prod_{n=1}^{N} e^{\log\left(1+\frac{1}{n}\right)-\frac{1}{n}}$$

$\Rightarrow$ for some $k\in\mathbb{Z}$,

$$-A + 2\pi i k = \lim_{N\to\infty}\sum_{n=1}^{N}\left[\log\left(\frac{n+1}{n}\right) - \frac{1}{n}\right]$$

$$= \lim_{N\to\infty}\left(\log\frac{2}{1}+\log\frac{3}{2}+\cdots+\log\frac{N}{N-1}+\log\frac{N+1}{N}\right)-\sum_{n=1}^{N}\frac{1}{n}$$

$$= -\lim_{N\to\infty}\left[\left(\sum_{n=1}^{N}\frac{1}{n}-\log N\right)-\log\left(1+\frac{1}{N}\right)\right]$$

$$= -\gamma$$

$$\therefore \quad \frac{1}{\Gamma(s)} = e^{(\gamma-2\pi i k)s}\cdot s\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}}, \qquad \forall s\in\mathbb{C}.$$

Putting $s=\frac{1}{2}$, (in fact, real not integer) we have $k=0$. ##