THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2040A Linear Algebra II, 1st Term, 2022-23
Test 1 (Date: 12th October 2022)

- Write down your Name and Student ID on the front page of answer book.
- It is not allowed to use any books or reference notes during the test.
- Give adequate explanation and justification for all your calculations and observations.
- Write your proofs in a clear and rigorous way.
- Answer all SEVEN questions. Total score: 100 points. Time allowed: 90 minutes
- **Convention:** To the end we discuss vector spaces over the only field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

1. *(10 points)* State without any proof the following two theorems that you learned from the lecture:
   (a) Replacement Theorem.
   (b) Dimension Theorem.

2. *(20 points)* Let V be a vector space and U_1, U_2 be subspaces of V.
   (a) Is $U_1 \cup U_2$ a subspace of V? Give reasons to your answer.
   (b) Prove that
   $$U_1 + U_2 := \{u_1 + u_2 : \ u_1 \in U_1, u_2 \in U_2\}$$
   is the smallest subspace of V that contains both U_1 and U_2.
   (c) Suppose $U_1 \cap U_2 = \{0\}$. Prove that for any $v \in U_1 + U_2$ it is unique to write v as $v = u_1 + u_2$ ($u_1 \in U_1, u_2 \in U_2$).

3. *(20 points)* Let $V = \{A \in M_{2\times 2}(\mathbb{C}) : \mathrm{Tr}(A) = 0\}$ denote the collection of all 2×2 complex matrices with trace zero.
   (a) Prove that V is a vector space over the real field $\mathbb{R}$ equipped with the usual addition and scalar multiplication of matrices.
   (b) Find a basis for the vector space V over $\mathbb{R}$.
   (c) Let $W = \{A = (a_{ij}) \in V : \ a_{21} = -\overline{a_{12}}\}$, where $\overline{a_{12}}$ denotes the complex conjugate of a_{12}. Prove that W is a subspace of V and further find a basis for W.

4. *(10 points)* Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \ldots, v_m + w$ is linearly dependent, then $w \in \mathrm{span}(\{v_1, \ldots, v_m\})$.

5. *(10 points)* Suppose v_1, v_2, v_3, v_4 is a basis for V. Show that
   $$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$
   is also a basis for V.

6. *(20 points)* Let
   $$U := \{p(x) \in P_5(\mathbb{R}) : p(-1) = p(0) = p(1) = 0\}.$$
   (a) Show that U is a subspace of $P_5(\mathbb{R})$.
   (b) Find a basis for U and determine the dimension for U.
   (c) Extend the basis of U in (b) to be a basis for $P_5(\mathbb{R})$.

7. *(10 points)* Suppose $T \in \mathcal{L}(V, W)$ for vector spaces V and W over the same field $\mathbb{F}$. Let $\{w_1, \ldots, w_m\}$ be a basis for range of T. Prove that there exist $f_1, \ldots, f_m \in \mathcal{L}(V, \mathbb{F})$ such that
   $$T(v) = f_1(v)w_1 + \cdots + f_m(v)w_m$$
   for every $v \in V$.

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