MATH2040 Midterm 1 Reference Solution

1. State without any proof the following two theorems that you learned from the lecture:
   (a) Replacement Theorem.
   (b) Dimension Theorem.

Solution:

(a) Let V be a vector space, $G \subseteq V$ be a finite spanning set with cardinality n, $L \subseteq V$ be a finite linearly independent subset with cardinality m. Then

   i. $m \leq n$

   ii. there exists $H \subseteq G$ with cardinality $n - m$ such that $V = \text{Span}(\ L \cup H\ )$

(b) Let V be a finite dimensional vector space, W be a vector space, $T \in \mathcal{L}(V, W)$. Then $\dim(V) = \text{rank}(T) + \text{nullity}(T)$

2. Let V be a vector space and U_1, U_2 be subspaces of V.
   (a) Is $U_1 \cup U_2$ a subspace of V? Give reasons to your answer.
   (b) Prove that
   $$U_1 + U_2 := \{u_1 + u_2 :\ u_1 \in U_1, u_2 \in U_2\}$$
   is the smallest subspace of V that contains both U_1 and U_2.
   (c) Suppose $U_1 \cap U_2 = \{0\}$. Prove that for any $v \in U_1 + U_2$ it is unique to write v as $v = u_1 + u_2$ $(u_1 \in U_1, u_2 \in U_2)$.

Solution:

(a) $U_1 \cup U_2$ may not be a subspace of V.

Consider $V = \mathbb{R}^2$ being the usual $\mathbb{R}^2$ plane, and $U_1 = \{\ (x, 0) \mid x \in \mathbb{R}\ \}$, $U_2 = \{\ (0, y) \mid y \in \mathbb{R}\ \}$. Then U_1, U_2 are subspaces of V but $U_1 \cup U_2$ is not.

(b) Since $0 \in U_1$ and $0 \in U_2$, we have $u_1 = u_1 + 0 \in U_1 + U_2$ for all $u_1 \in U_1$ and $u_2 = 0 + u_2 \in U_1 + U_2$ for all $u_2 \in U_2$, so $U_1 \subseteq U_1 + U_2$ and $U_2 \subseteq U_1 + U_2$.

Let $W \subseteq V$ be a subspace of V that contains both U_1 and U_2. Let $v \in U_1 + U_2$. Then there exists $u_1 \in U_1$ and $u_2 \in U_2$ such that $v = u_1 + u_2$. As $U_1 \subseteq W$ and $U_2 \subseteq W$, $u_1, u_2 \in W$. As W is a subspace, $v = u_1 + u_2 \in W$. As v is arbitrary, $U_1 + U_2 \subseteq W$.

(c) Let $v \in U_1 + U_2$, and $u_1, u_1' \in U_1$, $u_2, u_2' \in U_2$ be such that $v = u_1 + u_2$ and $v = u_1' + u_2'$. Then $u_1 + u_2 = v = u_1' + u_2'$ and so $u_1 - u_1' = u_2' - u_2$. Since U_1, U_2 are subspaces, by assumption we have $u_1 - u_1' \in U_1$ and $u_2' - u_2 \in U_2$. So $u_1 - u_1' = u_2' - u_2 \in U_1 \cap U_2$, which means that $u_1 - u_1' = u_2' - u_2 = 0$, $u_1 = u_1'$ and $u_2 = u_2'$. As v is arbitrary, for any $v \in U_1 + U_2$ it is unique to write v as $v = u_1 + u_2$ with $u_1 \in U_1$ and $u_2 \in U_2$.

3. Let $V = \{A \in M_{2\times 2}(\mathbb{C}) : \text{Tr}(A) = 0\}$ denote the collection of all 2×2 complex matrices with trace zero.
   (a) Prove that V is a vector space over the real field $\mathbb{R}$ equipped with the usual addition and scalar multiplication of matrices.
   (b) Find a basis for the vector space V over $\mathbb{R}$.
   (c) Let $W = \{A = (a_{ij}) \in V :\ a_{21} = -\overline{a_{12}}\}$, where $\overline{a_{12}}$ denotes the complex conjugate of a_{12}. Prove that W is a subspace of V and further find a basis for W.

Solution:

(a) Since the set $S = \{\ A \in M_{2\times 2}(\mathbb{C})\ \}$ is a vector space over $\mathbb{R}$ (with the usual addition and scalar multiplication), it suffices to show that V is a subspace of S.

Let $0_{2\times 2} \in M_{2\times 2}(\mathbb{C})$ be the zero matrix. Then $\text{Tr}(0_{2\times 2}) = 0 + 0 = 0$, so $0_{2\times 2} \in V$.

Let $A, B \in M_{2\times 2}(\mathbb{C})$. Then $\text{Tr}(A) = \text{Tr}(B) = 0$, so $\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B) = 0 + 0 = 0$, so $A + B \in V$.

Let $A \in M_{2\times 2}(\mathbb{C})$ and $r \in \mathbb{R}$. Then $\text{Tr}(A) = 0$, so $\text{Tr}(rA) = r\text{Tr}(A) = r \cdot 0 = 0$, so $rA \in V$.

As A, B, r are arbitrary, V is a subspace of the real vector space S, and hence is a real vector space itself.

(b) $\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} \right\}$ is a basis of V.

i. Let $r_1, \ldots, r_6 \in \mathbb{R}$ be such that

$$r_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + r_2 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + r_3 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + r_4 \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} + r_5 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + r_6 \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} = 0$$

Then $\begin{pmatrix} r_1 + ir_2 & r_3 + ir_4 \\ r_5 + ir_6 & -r_1 - ir_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. This implies that $r_1 = \ldots = r_6 = 0$. So β is linearly independent.

ii. Let $A \in V$. Then $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ for some $a_{11}, \ldots, a_{22} \in \mathbb{C}$. As $0 = \text{Tr}(A) = a_{11} + a_{22}$, we have $a_{22} = -a_{11}$. As $a_{11}, a_{12}, a_{21} \in \mathbb{C}$, there exists $r_1, \ldots, r_6 \in \mathbb{R}$ such that $a_{11} = r_1 + ir_2$, $a_{12} = r_3 + ir_4$, $a_{21} = r_5 + ir_6$. Hence $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix} = r_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + r_2 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + r_3 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + r_4 \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} + r_5 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + r_6 \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} \in$ Span(β). Since A is arbitrary, $V \subseteq$ Span(β)

It is easy to see that $B \in V$ for all $B \in \beta$, so $\beta \subseteq V$, Span(β) $\subseteq V$. So $V =$ Span(β).

Therefore β is a basis of V.

(c) i.
- Let $0_{2\times 2} \in V$ be the zero matrix. Then $a_{21} = 0 = -\overline{0} = -\overline{a_{12}}$, so $0_{2\times 2} \in W$.
- Let $A = (a_{ij}), B = (b_{ij}) \in W$. Then $a_{21} = -\overline{a_{12}}$ and $b_{21} = -\overline{b_{12}}$, so $(A + B)_{21} = (a_{21} + b_{21}) = -\overline{a_{12} + b_{12}} = -\overline{(A + B)_{12}}$. Hence $A + B \in W$.
- Let $A = (a_{ij}) \in W$ and $r \in \mathbb{R}$. Then $a_{21} = -\overline{a_{12}}$, so $(rA)_{21} = ra_{21} = -r\overline{a_{12}} = -\overline{ra_{21}} = -\overline{(rA)_{21}}$. Thus $rA \in W$.

So W is a subspace of V.

ii. $\gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\}$ is a basis for W. Other bases are of course accepted

iii.
- Let $r_1, \ldots, r_4 \in \mathbb{R}$ be such that $r_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + r_2 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + r_3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + r_4 \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = 0$. Then $\begin{pmatrix} r_1 + ir_2 & r_3 + ir_4 \\ -r_3 + ir_4 & -(r_1 + ir_2) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and so $r_1 = \ldots = r_4 = 0$. This implies that γ is linearly independent.
- Let $A = (a_{ij}) \in W$. Then $a_{22} = -a_{11}$ and $a_{21} = -\overline{a_{12}}$. As $a_{11}, a_{12} \in \mathbb{C}$, there exists $r_1, \ldots, r_4 \in \mathbb{R}$ such that $a_{11} = r_1 + ir_2$ and $r_{12} = r_3 + ir_4$. So $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} r_1 + ir_2 & r_3 + ir_4 \\ -r_3 + ir_4 & -r_1 - ir_2 \end{pmatrix} = r_1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + r_2 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + r_3 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + r_4 \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in$ Span(γ). As A is arbitrary, $W \subseteq$ Span(γ).

  It is easy to see that $A \in W$ for all $A \in \gamma$, so $\gamma \subseteq W$, Span(γ) $\subseteq W$. So $W =$ Span(γ).

Therefore γ is a basis of W.

4. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \ldots, v_m + w$ is linearly dependent, then $w \in \text{span}(\{v_1, \ldots, v_m\})$.

Solution:

As $v_1 + w, \ldots, v_m + w$ is linearly dependent, there exists scalars $a_1, \ldots, a_m$ not all zero such $a_1(v_1 + w) + \ldots + a_m(v_m + w) = 0$, so $a_1v_1 + \ldots + a_mv_m = -(a_1 + \ldots + a_m)w$. If $a_1 + \ldots + a_m = 0$, we would have $a_1v_1 + \ldots + a_mv_m = 0$ with $a_1, \ldots, a_n$ not all zero. This would contradict the assumption that $v_1, \ldots, v_m$ is linearly independent, so $a_1 + \ldots + a_m \neq 0$. Hence $w = -\frac{a_1}{a_1 + \ldots + a_m} v_1 - \ldots - -\frac{a_m}{a_1 + \ldots + a_m} v_m \in \text{Span}(\ \{ v_1, \ldots, v_m \}\ )$.

5. Suppose v_1, v_2, v_3, v_4 is a basis for V. Show that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis for V.

Solution:

As $\{ v_1, \ldots, v_4 \}$ is a basis, it is easy to see $v_1+v_2, v_2+v_3, v_3+v_4, v_4$ are distinct, so $|\{ v_1+v_2, v_2+v_3, v_3+v_4, v_4 \}| = 4 = |\{ v_1, \ldots, v_4 \}| = \dim(V)$. So to show that $\{ v_1+v_2, v_2+v_3, v_3+v_4, v_4 \}$ is a basis, it suffices to show its linear independence.

Let $a_1, \ldots, a_4$ be scalars such that $a_1(v_1+v_2)+a_2(v_2+v_3)+a_3(v_3+v_4)+a_4v_4 = 0$. Then $a_1v_1+(a_1+a_2)v_2+(a_2+a_3)v_3+(a_3+a_4)v_4 = 0$. As $\{ v_1, \ldots, v_4 \}$ is a basis, it is linearly independent, so $a_1 = a_1+a_2 = a_2+a_3 = a_3+a_4 = 0$. Hence $a_1 = \ldots = a_4 = 0$. This implies that $\{ v_1+v_2, v_2+v_3, v_3+v_4, v_4 \}$ is linearly independent and so is a basis.

6. Let

$$U := \{p(x) \in P_5(\mathbb{R}) : p(-1) = p(0) = p(1) = 0\}.$$

(a) Show that U is a subspace of $P_5(\mathbb{R})$.

(b) Find a basis for U and determine the dimension for U.

(c) Extend the basis of U in (b) to be a basis for $P_5(\mathbb{R})$.

Solution:

(a)
- Let $0(x) \in \mathsf{P}_5(\mathbb{R})$ be the zero polynomial. Then $0(-1) = 0(0) = 0(1) = 0$, so $0(x) \in U$.
- Let $p(x), q(x) \in U$. Then $p(-1) = p(0) = p(1) = q(-1) = q(0) = q(1) = 0$, so $(p+q)(-1) = p(-1)+q(-1) = 0$, $(p+q)(0) = p(0)+q(0) = 0$, $(p+q)(1) = p(1)+q(1) = 0$. Also $p(x)+q(x) \in \mathsf{P}_5(\mathbb{R})$, hence $p(x)+q(x)\ inU$.
- Let $p(x) \in U$, $c \in \mathbb{R}$. Then $p(-1) = p(0) = p(1) = 0$, so $(cp)(-1) = cp(-1) = 0$, $(cp)(0) = cp(0) = 0$, $(cp)(1) = cp(1) = 0$. Also, $cp(x) \in \mathsf{P}_5(\mathbb{R})$, so $cp(x) \in U$.

Hence U is a subspace of $\mathsf{P}_5(\mathbb{R})$.

(b)
i. $\beta = \left\{ x(x-1)(x+1), x^2(x-1)(x+1), x^3(x-1)(x+1) \right\}$ is a basis of U

ii.
- Since the elements of β are of distinct degree, β is linearly independent
- Let $p(x) \in U$. As $p(-1) = p(0) = p(1) = 0$, by factor theorem there exists a polynomial $q(x) \in \mathsf{P}(\mathbb{R})$ such that $p(x) = x(x-1)(x+1)q(x)$. Since $p(x) \in \mathsf{P}_5(\mathbb{R})$, $\deg p(x) \leq 5$. This implies that $\deg q(x) \leq 2$ and so $q(x) = a+bx+cx^2$ for some $a, b, c \in \mathbb{R}$. Hence $p(x) = x(x-1)(x+1)q(x) = x(x-1)(x+1)(a+bx+cx^2) = ax(x-1)(x+1)+bx^2(x-1)(x+1)+cx^3(x-1)(x+1) \in \mathrm{Span}(\ \beta\ )$. As $p(x)$ is arbitrary, $U \subseteq \mathrm{Span}(\ \beta\ )$. As $\beta \subseteq U$, we have $\mathrm{Span}(\ \beta\ ) \subseteq U$ and so $U = \mathrm{Span}(\ \beta\ )$.

This implies that β is a basis of U.

iii. Since β is a basis of U, $\dim(U) = |\ \beta\ | = 3$.

(c) Let $\gamma = \beta \cup \{ 1, x, x^2 \} \subseteq \mathsf{P}_5(\mathbb{R})$. Then $|\ \gamma\ | = 6 = \dim(\mathsf{P}_5(\mathbb{R}))$. So to show that γ is a basis of $\mathsf{P}_5(\mathbb{R})$ it suffices to show its linear independence.

Since elements of γ are of distinct degree, γ is linearly independent.

So γ is a basis of $\mathsf{P}_5(\mathbb{R})$ that extends β.

7. Suppose $T \in \mathcal{L}(V, W)$ for vector spaces V and W over the same field $\mathbb{F}$. Let $\{w_1, \ldots, w_m\}$ be a basis for range of T. Prove that there exist $f_1, \ldots, f_m \in \mathcal{L}(V, \mathbb{F})$ such that

$$T(v) = f_1(v)w_1 + \cdots + f_m(v)w_m$$

for every $v \in V$.

Solution:

For each $v \in V$ define $f_1(v), \ldots, f_m(v) \in \mathbb{F}$ be the scalars such that

$$T(v) = f_1(v)w_1 + \ldots + f_m(v)w_m$$

Such scalars exist and are unique since $T(v) \in \mathsf{R}(\ T\ )$ and $\{ w_1, \ldots, w_m \}$ is a basis of $\mathsf{R}(\ T\ )$.

Let $v_1, v_2 \in V$. Then $T(v_1) = f_1(v_1)w_1 + \ldots + f_m(v_1)w_m$, $T(v_2) = f_1(v_2)w_1 + \ldots + f_m(v_2)w_m$. Then $f_1(v_1+v_2)w_1 + \ldots + f_m(v_1+v_2)w_m = T(v_1+v_2) = T(v_1)+T(v_2) = (f_1(v_1)+f_1(v_2))w_1 + \ldots + (f_m(v_1)+f_m(v_2))w_m$. By the uniqueness of the scalars, $f_i(v_1+v_2) = f_i(v_1)+f_i(v_2)$ for all $i \in \{ 1, \ldots, m \}$.

Let $v \in V$ and $c \in \mathbb{F}$. Then $T(v) = f_1(v)w_1 + \ldots + f_m(v)w_m$. So $f_1(cv)w_1 + \ldots + f_m(cv)w_m = T(cv) = cT(v) = (cf_1(v))w_1 + \ldots + (cf_m(v))w_m$. By the uniqueness of the scalars, $f_i(cv) = cf_i(v)$ for all $i \in \{ 1, \ldots, m \}$.

Hence $f_i \in \mathcal{L}(V, \mathbb{F})$ for all $i \in \{ 1, \ldots, m \}$.