THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2040A
Solution to Homework 9

Compulsory Part

Sec. 6.2

Q2(g).

Sol. By direct calculation, the basis is

$$\left\{\frac{1}{6}\begin{pmatrix}3 & 5\\ -1 & 1\end{pmatrix}, \frac{1}{3\sqrt{2}}\begin{pmatrix}-2 & 2\\ 3 & -1\end{pmatrix}, \frac{1}{3\sqrt{2}}\begin{pmatrix}3 & -1\\ 2 & -2\end{pmatrix}\right\}.$$

The coefficients are 24, $6\sqrt{2}$, $-9\sqrt{2}$.

Q6. Let V be an inner product space, and let W be a finite-dimensional subspace of V. If $x \notin W$, prove that there exists $y \in V$ such that $y \in W^\perp$, but $\langle x, y\rangle \neq 0$. Hint: Use Theorem 6.6.

Sol. By Theorem 6.6, there exists $w \in W$ and $y \in W^\perp$ such that $x = w + y$. Since $x \notin W$, $y \neq \overrightarrow{0}$. Then we have

$$\langle x, y\rangle = \langle w + y, y\rangle = \langle w, y\rangle + \langle y, y\rangle = ||y||^2 > 0.$$

Q10. Let W be a finite-dimensional subspace of an inner product space V. Prove that there exists a projection T on W along $W^\perp$ that satisfies $\mathsf{N}(T) = W^\perp$. In addition, prove that $||T(x)|| \leq ||x||$ for all $x \in V$. Hint: Use Theorem 6.6 and Exercise 10 of Section 6.1. (Projections are defined in the exercises of Section 2.1.)

Sol. By Q13(d) $V = W \oplus W^\perp$. Hence there exists a projection T on W along $W^\perp$ that satisfies $\mathsf{N}(T) = W^\perp$. (See Solution to HW5, Sec 2.3 Q17.)

For any $x \in V$, there exists unique $w \in W$ and $y \in W^\perp$ such that $x = w + y$. Then $T(x) = w$ and by Sec 6.1 Q10 we have $||x||^2 = ||w||^2 + ||y||^2$. Therefore $||x||^2 \geq ||T(x)||^2$. Since $||x||, ||T(x)|| \geq 0$, we have $||x|| \geq ||T(x)||$.

Q13. Let V be an inner product space, S and S_0 be subsets of V, and W be a finite-dimensional subspace of V. Prove the following results.

(a) $S_0 \subseteq S$ implies that $S^\perp \subseteq S_0^\perp$.
(b) $S \subseteq (S^\perp)^\perp$; so $\text{span}(S) \subseteq (S^\perp)^\perp$.
(c) $W = (W^\perp)^\perp$. Hint: Use Exercise 6.
(d) $V = W \oplus W^\perp$. (See the exercises of Section 1.3.)

Sol. (a) If $v \in S^\perp$, then $\langle v, s\rangle = 0$ for all $s \in S$. In particular $\langle v, s\rangle = 0$ for all $s \in S_0$. Therefore $v \in S_0^\perp$.

(b) Let $v \in S$. For all $u \in S^\perp$, $\langle u, v\rangle = 0$. Hence $\langle v, u\rangle = \overline{\langle u, v\rangle} = 0$ and $v \in (S^\perp)^\perp$. Since $(S^\perp)^\perp$ is a subspace of V containing S, $\text{span}(S) \subseteq (S^\perp)^\perp$.

(c) By part (b), $W \subset (W^\perp)^\perp$. If $x \notin W$, by Q6., there exists $y \in W^\perp$ such that $\langle x, y\rangle \neq 0$. Therefore $x \notin (W^\perp)^\perp$.

(d) By Theorem 6.6, we have $V = W + W^\perp$. Also, from its proof, we have $W \cap W^\perp = \{\overrightarrow{0}\}$. Therefore $V = W \oplus W^\perp$.

Sec. 6.3

Q2(c). For each of the following inner product spaces V (over F) and linear transformations $g : V \to F$, find a vector y such that $g(x) = \langle x, y\rangle$ for all $x \in V$.

$$V = P_2(\mathbb{R}) \text{ with } \langle f, h\rangle = \int_0^1 f(t)h(t)dt,\ g(f) = f(0) + f'(1).$$

Sol. Let $\beta = \{1, x, x^2\}$ be the standard basis of V. Such y exists by Theorem 6.8 and so we write $y = a_1 + a_2x + a_3x^2$ for some $a_1, a_2, a_3 \in \mathbb{R}$. Then for all $f = b_1 + b_2x + b_3x^2 \in V$ with $b_1, b_2, b_3 \in \mathbb{R}$, we have

$$g(f) = b_1+b_2+2b_3 \text{ and } \langle f, y\rangle = a_1b_1+\frac{1}{2}(a_1b_2+a_2b_1)+\frac{1}{3}(a_1b_3+a_2b_2+a_3b_1)+\frac{1}{4}(a_2b_3+a_3b_2)+\frac{1}{5}(a_3b_3).$$

Since b_1, b_2, b_3 are arbitrary, the coefficients of them on both sides of $g(f) = \langle f, y\rangle$ must equal respectively. Therefore we can summarize as

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

By computation, we have $a_1 = 33$, $a_2 = -204$, $a_3 = 210$. Hence the desired y is given by $210x^2 - 204x + 33$.

Q3(c). For each of the following inner product spaces V and linear operators T on V, evaluate T^* at the given vector in V.

$$V = P_1(\mathbb{R}) \text{ with } \langle f, g\rangle = \int_{-1}^1 f(t)g(t)dt,\ T(f) = f' + 3f.\ f(t) = 4 - 2t.$$

Sol. Let $\beta = \{1, t\}$ be the standard basis of V. Write $T^*(4 - 2t) = a + bt$ for some $a, b \in \mathbb{R}$. Then for any $g(t) = c + dt \in V$ with $c, d \in \mathbb{R}$, we have $T(g(t)) = d + 3c + 3dt$ and

$$\langle d + 3c + 3dt, 4 - 2t\rangle = \langle T(g(t), 4 - 2t\rangle = \langle g(t), T^*(4 - 2t)\rangle = \langle c + dt, a + bt\rangle.$$

Now $\langle d + 3c + 3dt, 4 - 2t\rangle = 2(4)(d + 3c) + (3d)(-2)\frac{2}{3} = 4d + 24c$ and $\langle c + dt, a + bt\rangle = 2ac + \frac{2}{3}bd$. Since c, d are arbitrary, the coefficients of them on both sides of the equation must equal respectively. Therefore $24 = 2a$ and $\frac{2}{3}b = 4$. Hence $a = 12$ and $b = 6$. So $T^*(4 - 2t) = 12 + 6t$.

Q12. Let V be an inner product space, and let T be a linear operator on V. Prove the following results.

i. $\mathsf{R}(T^*)^\perp = \mathsf{N}(T)$.

ii. If V is finite-dimensional, then $\mathsf{R}(T^*) = \mathsf{N}(T)^\perp$. Hint: Use Exercise 13(c) of Section 6.2.

Sol. i. Let $x \in \mathsf{R}(T^*)^\perp$. Then we have

$$0 = \langle x, T^*(T(x)) \rangle = \langle T(x), T(x) \rangle = ||T(x)||^2.$$

Hence $T(x) = \overrightarrow{0}$ and $x \in \mathsf{N}(T)$.

Conversely, suppose $x \in \mathsf{N}(T)$. For all $z \in \mathsf{R}(T^*)$, there exists $y \in V$ such that $z = T^*(y)$. Hence

$$\langle x, z \rangle = \langle x, T^*(y) \rangle = \langle T(x), y \rangle = 0$$

and $x \in \mathsf{R}(T^*)^\perp$.

ii. If V is finite dimensional, by Q13(c) of Sec. 6.2, $\mathsf{N}(T)^\perp = (\mathsf{R}(T^*)^\perp)^\perp = \mathsf{R}(T^*)$.

Q14. Let V be an inner product space, and let $y, z \in V$. Define $T : V \to V$ by $T(x) = \langle x, y \rangle z$ for all $x \in V$. First prove that T is linear. Then show that T^* exists, and find an explicit expression for it.

Sol. For all $x, w \in V$, we have

$$\langle T(x), w \rangle = \langle \langle x, y \rangle z, w \rangle = \langle x, y \rangle \langle z, w \rangle = \left\langle x, \overline{\langle z, w \rangle} y \right\rangle = \langle x, \langle w, z \rangle y \rangle .$$

Note that $w \mapsto \langle w, z \rangle y$ is a linear operator on V since

$$\langle w_1 + cw_2, z \rangle y = (\langle w_1, z \rangle + c \langle w_2, z \rangle) y = \langle w_1, z \rangle y + c \langle w_2, z \rangle y$$

for all w_1, $w_2 \in V$ and scalar c. Therefore this gives the adjoint of T.

Optional Part

Sec. 6.2

Q1. Label the following statements as true or false.

i. The Gram-Schmidt orthogonalization process allows us to construct an orthonormal set from an arbitrary set of vectors.

ii. Every nonzero finite-dimensional inner product space has an orthonormal basis.

iii. The orthogonal complement of any set is a subspace.

iv. If $\{v_1, v_2, \ldots, v_n\}$ is a basis for an inner product space V, then for any $x \in V$ the scalars $\langle x, v_i \rangle$ are the Fourier coefficients of x.

v. An orthonormal basis must be an ordered basis.

vi. Every orthogonal set is linearly independent.

vii. Every orthonormal set is linearly independent.

Sol. i. False. Consider $\{\overrightarrow{0}\}$.

ii. True.

iii. True.

iv. False. The notion of Fourier coefficients is only defined for orthonormal basis.

v. True. This is by definition.

vi. False. Consider $\{\overrightarrow{0}\}$.

vii. True.

Q2(i).

Sol. The basis is

$$\left\{\sqrt{\frac{2}{\pi}}\sin t, \sqrt{\frac{2}{\pi}}\cos t, \frac{\pi - 4\sin t}{\sqrt{\pi^3 - 8\pi}}, \frac{8\cos t + 2\pi t - \pi^2}{\sqrt{\frac{\pi^5}{3} - 32\pi}}\right\}.$$

The coefficients are $\sqrt{\frac{2}{\pi}}(2\pi+2)$, $-4\sqrt{\frac{2}{\pi}}$, $\frac{\pi^3+\pi^2-8\pi-8}{\sqrt{\pi^3-8\pi}}$, $\frac{\pi^4-96}{\sqrt{3(\pi^5-96\pi)}}$.

Q4. Let $S = \{(1,0,i),(1,2,1)\}$ in $\mathbb{C}^3$. Compute $S^\perp$.

Sol. $S^\perp = \{(a,b,c) \in \mathbb{C}^3 : \langle (a,b,c),(1,0,i)\rangle = a - ic = 0$ and $\langle (a,b,c),(1,2,1)\rangle = a+2b+c = 0\}$. Therefore we would like to solve

$$\begin{pmatrix} 1 & 0 & -i \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

for $(a,b,c) \in \mathbb{C}^3$. By computation

$$\begin{pmatrix} 1 & 0 & -i \\ 1 & 2 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -i \\ 0 & 1 & \frac{1+i}{2} \end{pmatrix}.$$

Therefore the null space is spanned by $\begin{pmatrix} i \\ -\frac{1+i}{2} \\ 1 \end{pmatrix}$ and

$$S^\perp = \mathrm{span}(\{(i, -\frac{1+i}{2}, 1\}).$$

Q14. Let W_1 and W_2 be subspaces of a finite-dimensional inner product space. Prove that $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$ and $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$. (See the definition of the sum of subsets of a vector space on page 22.) Hint for the second equation: Apply Exercise 13(c) to the first equation.

Sol. Since $W_1, W_2 \subset W_1 + W_2$, by Q13(a), $(W_1+W_2)^\perp$ is contained in $W_1^\perp$ and $W_2^\perp$. Therefore $(W_1+W_2)^\perp \subset W_1^\perp \cap W_2^\perp$.

On the other hand, if $x \in W_1^\perp \cap W_2^\perp$, for all $w \in W_1 + W_2$, there exists $w_1 \in W_1$, $w_2 \in W_2$, such that $w = w_1 + w_2$. Since $\langle x, w_1\rangle = \langle x, w_2\rangle = 0$, we have $\langle x, w\rangle = \langle x, w_1\rangle + \langle x, w_2\rangle = 0$. Therefore $x \in (W_1+W_2)^\perp$ and hence $(W_1+W_2)^\perp = W_1^\perp \cap W_2^\perp$.

By applying this with W_1 and W_2 replaced by $W_1^\perp$ and $W_2^\perp$ respectively, and applying Q13(c), we have $(W_1^\perp + W_2^\perp)^\perp = (W_1^\perp)^\perp \cap (W_2^\perp)^\perp = W_1 \cap W_2$. By taking orthogonal complement on both sides and applying Q13(c) again, we have $(W_1 \cap W_2)^\perp = ((W_1^\perp + W_2^\perp)^\perp)^\perp = W_1^\perp + W_2^\perp$.

Q17. Let T be a linear operator on an inner product space V. If $\langle T(x), y\rangle = 0$ for all $x, y \in V$, prove that $T = T_0$. In fact, prove this result if the equality holds for all x and y in some basis for V.

Sol. For all $x \in V$, $T(x) \in V$ and thus $||T(x)||^2 = \langle T(x), T(x) \rangle = 0$ by taking $y = T(x)$. Hence $T(x) = \overrightarrow{0}$ for all $x \in V$ and $T = T_0$ the zero transformation.

Now we suppose $\langle T(x), y \rangle = 0$ for all x and y in some basis β for V. We want to prove that this implies $\langle T(x'), y' \rangle = 0$ for all x' and y' in V.

Since β is a basis, there exists $x_1, \ldots, x_m \in \beta$, $y_1, \ldots, y_n \in \beta$, and scalars $a_1, \ldots, a_m, b_1, \ldots, b_n$ such that

$$x' = \sum_{i=1}^{m} a_i x_i \text{ and } y' = \sum_{j=1}^{n} b_j y_j.$$

Then we have

$$\begin{aligned}
&\langle T(x'), y' \rangle \\
=&\left\langle T\left(\sum_{i=1}^{m} a_i x_i\right), \sum_{j=1}^{n} b_j y_j \right\rangle \\
=&\left\langle \sum_{i=1}^{m} a_i T(x_i), \sum_{j=1}^{n} b_j y_j \right\rangle \\
=&\sum_{i=1}^{m}\sum_{j=1}^{n} a_i \overline{b_j} \langle T(x_i), y_j \rangle \\
=&0.
\end{aligned}$$

Q18. Let $V = \mathsf{C}([-1, 1])$. Suppose that W_e and W_o denote the subspaces of V consisting of the even and odd functions, respectively. (See Exercise 22 of Section 1.3.) Prove that $W_e^{\perp} = W_o$, where the inner product on V is defined by

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)dt.$$

Sol. Let $f \in W_o$. For any $g \in W_e$, $h(t) := f(t)g(t)$ for all $t \in [-1, 1]$ is an odd function since $h(-t) = f(-t)g(-t) = -f(t)g(t) = -h(t)$. Therefore

$$\langle f, g \rangle = \int_{-1}^{1} f(t)g(t)dt = 0.$$

Hence $W_o \subset W_e^{\perp}$.

Note that $V = W_e \oplus W_o$. In fact, let $h \in V$. Define

$$h_e(t) := \frac{1}{2}(h(t) + h(-t)) \text{ and } h_o(t) := \frac{1}{2}(h(t) - h(-t)) \; \forall t \in [-1, 1].$$

Then $h_e \in W_e$, $h_o \in W_o$, and $h = h_e + h_o$. Hence $V = W_e + W_o$. Moreover, if $h \in W_e \cap W_o$, then $h(t) = h(-t) = -h(t)$ for all $t \in [-1, 1]$. Therefore $2h(t) = 0$ and $h(t) = 0$ for all $t \in [-1, 1]$. So $W_e \cap W_o = \{\overrightarrow{0}\}$.

Let $f \in W_e^{\perp}$. Since $V = W_e \oplus W_o$, write $f = f_e + f_o$ with $f_e \in W_e$ and $f_o \in W_o$. Then

$$0 = \langle f, f_e \rangle = \langle f_o + f_e, f_e \rangle = \langle f_o, f_e \rangle + \langle f_e, f_e \rangle = ||f_e||^2.$$

Therefore $f_e = 0$ and $f = f_o \in W_o$. So we have the opposition inclusion and $W_e^{\perp} = W_o$.

Sec. 6.3

Q1. Label the following statements as true or false. Assume that the underlying inner product spaces are finite-dimensional.

(a) Every linear operator has an adjoint.

(b) Every linear operator on V has the form $x \to \langle x, y \rangle$ for some $y \in V$.

(c) For every linear operator T on V and every ordered basis β for V, we have $[T^*]_\beta = ([T]_\beta)^*$.

(d) The adjoint of a linear operator is unique.

(e) For any linear operators T and U and scalars a and b,

$$(aT + bU)^* = aT^* + bU^*.$$

(f) For any $n \times n$ matrix A, we have $(L_A)^* = L_{A^*}$.

(g) For any linear operator T, we have $(T^*)^* = T$.

Sol. (a) True. (Remark: Here the finiteness of dimension of the underlying inner product space is essential.)

(b) False. If $V \neq \mathbb{R}$, then the codomain of any linear operator on V cannot be $\mathbb{R}$.

(c) False.

(d) True.

(e) False. Over complex inner product space, $(iI_V)^* = -iI_V \neq iI_V$.

(f) True.

(g) True.

Q9. Prove that if $V = W \oplus W^\perp$ and T is the projection on W along $W^\perp$, then $T = T^*$. Hint: Recall that $\mathsf{N}(T) = W^\perp$. (For definitions, see the exercises of Sections 1.3 and 2.1.)

Sol. From the assumption $V = W \oplus W^\perp$, for all $v, w \in V$, there exist unique $v_1, w_1 \in W$ and $v_2, w_2 \in W^\perp$ such that $v = v_1 + v_2$ and $w = w_1 + w_2$. We check that

$$\langle T(v), w \rangle = \langle v_1, w_1 + w_2 \rangle = \langle v_1, w_1 \rangle + \langle v_1, w_2 \rangle = \langle v_1, w_1 \rangle$$

and so

$$\langle v, T(w) \rangle = \overline{\langle T(w), v \rangle} = \overline{\langle w_1, v_1 \rangle} = \langle v_1, w_1 \rangle = \langle T(v), w \rangle .$$

Therefore T^* exists and $T = T^*$.

Q10. Let T be a linear operator on an inner product space V. Prove that $||T(x)|| = ||x||$ for all $x \in V$ if and only if $\langle T(x), T(y) \rangle = \langle x, y \rangle$ for all $x, y \in V$. Hint: Use Exercise 20 of Section 6.1.

Sol. ($\Leftarrow$) Suppose $\langle T(x), T(y) \rangle = \langle x, y \rangle$ for all $x, y \in V$. Then $||T(x)|| = \sqrt{\langle T(x), T(x) \rangle} = \sqrt{\langle x, x \rangle} = ||x||$ for all $x \in V$.

(⇒) Suppose $||T(x)|| = ||x||$ for all $x \in V$. Then $||T(x)||^2 = ||x||^2$ for all $x \in V$. If V is a real inner product space, by Exercise 20 of Section 6.1, we have

$$\langle T(x), T(y)\rangle = \frac{1}{4}||T(x)+T(y)||^2 - \frac{1}{4}||T(x)-T(y)||^2 = \frac{1}{4}||T(x+y)||^2 - \frac{1}{4}||T(x-y)||^2$$
$$= \frac{1}{4}||x+y||^2 - \frac{1}{4}||x-y||^2 = \langle x, y\rangle .$$

If V is a complex inner product space, by Exercise 20 of Section 6.1, we have

$$\langle T(x), T(y)\rangle = \frac{1}{4}\sum_{k=1}^{4} ||T(x)+i^kT(y)||^2 = \frac{1}{4}\sum_{k=1}^{4} ||T(x+i^ky)||^2 = \frac{1}{4}\sum_{k=1}^{4} ||x+i^ky||^2 = \langle x, y\rangle .$$

Q11. For a linear operator T on an inner product space V. Prove that $T^*T = T_0$ implies $T = T_0$. Is the same result true if we assume that $TT^* = T_0$?

Sol. Suppose $T^*T = T_0$. Let $x \in V$. Note that

$$||T(x)||^2 = \langle T(x), T(x)\rangle = \langle x, T^*T(x)\rangle = \langle x, \overrightarrow{0}\rangle = 0.$$

Hence $T(x) = \overrightarrow{0}$ and thus $T = T_0$.

Suppose now $TT^* = T_0$. Since T^* exists, $(T^*)^*$ exists and equals to T. So by previous argument, $T^* = T_0$. The adjoint of the zero operator is still zero since $\langle T_0(x), y\rangle = \langle x, T_0(y)\rangle$ for all $x, y \in V$. Therefore $T = (T^*)^* = T_0^* = T_0$.

Q13. Let T be a linear operator on a finite-dimensional inner product space V. Prove the following results.

(a) $\mathsf{N}(T^*T) = \mathsf{N}(T)$. Deduce that $\text{rank}(T^*T) = \text{rank}(T)$.

(b) $\text{rank}(T) = \text{rank}(T^*)$. Deduce from (a) that $\text{rank}(TT^*) = \text{rank}(T)$.

(c) For any $n \times n$ matrix A. $\text{rank}(A^*A) = \text{rank}(AA^*) = \text{rank}(A)$.

Sol. (a) It is clear that $\mathsf{N}(T) \subset \mathsf{N}(T^*T)$. Let $x \in \mathsf{N}(T^*T)$. Then $\langle T(x), T(x)\rangle = \langle x, T^*T(x)\rangle = \langle x, \overrightarrow{0}\rangle = 0$. Hence $T(x) = \overrightarrow{0}$ and $x \in \mathsf{N}(T)$. It follows that

$$\text{rank}(T^*T) = n - \text{nullity}(T^*T) = n - \text{nullity}(T) = \text{rank}(T)$$

where $n = \dim(V)$.

(b) By Q12(b), $\mathsf{R}(T^*) = \mathsf{N}(T)^\perp$. Since $V = \mathsf{N}(T) \oplus \mathsf{N}(T)^\perp$ by Sec 6.2 Q13(d), we have $n = \text{nullity}(T) + \dim(\mathsf{N}(T)^\perp)$ and

$$\text{rank}(T^*) = \dim(\mathsf{N}(T)^\perp) = n - \text{nullity}(T) = \text{rank}(T).$$

(c) Note that $L_A^* = L_{A^*}$. Hence by applying part (a) and (b) with $T = L_A$, we have $\text{rank}(A^*A) = \text{rank}(L_{A^*}L_A) = \text{rank}(L_A^*L_A) = \text{rank}(L_A) = \text{rank}(A)$. Similarly, $\text{rank}(AA^*) = \text{rank}(A)$.