THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2040A/B (First Term, 2018-19)
Linear Algebra II
Solution to Homework 12

Compulsory Part

Sec. 6.4

2 Q: For each linear operator T on an inner product space V, determine whether T is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of T for V and list the corresponding eigenvalues.

(d) $V = \mathsf{P}_2(\mathbb{R})$ and T is defined by $T(f) = f'$, where

$$\langle f, g\rangle = \int_0^1 f(t)g(t)dt.$$

Sol: (d) We first obtain the orthonormal basis $\beta = \{e_1, e_2, e_3\}$ for V by performing Gram-Schmidt Orthogonalization Process on the basis $\{1, x, x^2\}$ for V, where

$$e_1 = 1, \quad e_2 = 2\sqrt{3}(x - \tfrac{1}{2}), \quad e_3 = 6\sqrt{5}(x^2 - x + \tfrac{1}{6}).$$

Note that $T(e_1) = 0$, $T(e_2) = 2\sqrt{3}e_1$ and $T(e_3) = 2\sqrt{15}e_2$. Then

$$[T]_\beta = \begin{pmatrix} 0 & 2\sqrt{3} & 0 \\ 0 & 0 & 2\sqrt{15} \\ 0 & 0 & 0 \end{pmatrix}.$$

Clearly, $[T^*]_\beta = [T]_\beta^* \neq [T]_\beta$. Hence T is not self-adjoint. We see that the $(1,1)$-entry of $[T^*T]_\beta = [T]_\beta^*[T]_\beta$ is 0 while that of $[TT^*]_\beta = [T]_\beta[T]_\beta^*$ is $(2\sqrt{3})^2 = 12 \neq 0$. Therefore, T is also not normal. Then there is no orthonormal basis of eigenvectors of T for V.

7 Q: Let T be a linear operator on an inner product space V, and let W be a T-invariant subspace of V. Prove the following results.

(a) If T is self-adjoint, then T_W is self-adjoint.

(b) $W^\perp$ is T^*-invariant.

(c) If W is both T- and T^*-invariant, then $(T_W)^* = (T^*)_W$.

(d) If W is both T- and T^*-invariant and T is normal, then T_W is normal.

Sol: (a) $\forall u, v \in W$, since T is self-adjoint,

$$\langle T_W(u), v\rangle = \langle T(u), v\rangle = \langle u, T(v)\rangle = \langle u, T_W(v)\rangle,$$

whence T_W is self-adjoint.

(b) Fix $w' \in W^\perp$ and $w \in W$. As W is T-invariant, $T(w) \in W$. Then

$$\langle w, T^*(w')\rangle = \langle T(w), w'\rangle = 0.$$

Therefore, $T^*(w) \in W^\perp$. $W^\perp$ is T^*-invariant.

(c) Fix $w \in W$. We claim that $(T_W)^*(w) = (T^*)_W(w)$. If suffices to show that $\forall w' \in W$, $\langle w', (T_W)^*(w)\rangle = \langle w', (T^*)_W(w)\rangle$. Indeed, $\forall w' \in W$,

$$\langle w', (T_W)^*(w)\rangle = \langle T_W(w'), w\rangle = \langle T(w'), w\rangle = \langle w', T^*(w)\rangle = \langle w', (T^*)_W(w)\rangle.$$

Therefore, $(T_W)^* = (T^*)_W$.

(d) We have $T_W(T_W)^* = T_W(T^*)_W = (TT^*)_W = (T^*T)_W = (T^*)_W T_W = (T_W)^* T_W$. Therefore, T_W is normal.

9 Q: Let T be a normal operator on a finite-dimensional inner product space V. Prove that $\mathsf{N}(T) = \mathsf{N}(T^*)$ and $\mathsf{R}(T) = \mathsf{R}(T^*)$.

Sol: Fix $v \in \mathsf{N}(T)$. If $v = \vec{0}$, then clearly $v \in \mathsf{N}(T^*)$. If $v \neq \vec{0}$, then v is an eigenvector of T corresponding to eigenvalue 0 and by Theorem 6.15, v is also an eigenvector of T^* corresponding to eigenvalue $\overline{0} = 0$, implying that $v \in \mathsf{N}(T^*)$. We have $\mathsf{N}(T) \subset \mathsf{N}(T^*)$. Note that T^* is also normal. Applying the above argument on T^* yields $\mathsf{N}(T^*) \subset \mathsf{N}((T^*)^*) = \mathsf{N}(T)$. Hence, $\mathsf{N}(T) = \mathsf{N}(T^*)$.
By Exercise 12 in Sec. 6.3, $\mathsf{R}(T^*) = \mathsf{N}(T)^\perp = \mathsf{N}(T^*)^\perp = \mathsf{R}((T^*)^*) = \mathsf{R}(T)$.

Sec. 6.5

2 Q: For each of the following matrices A, find an orthogonal or unitary matrix P and a diagonal matrix D such that $P^*AP = D$.

(c)

$$\begin{pmatrix} 2 & 3-3i \\ 3+3i & 5 \end{pmatrix}$$

Sol: (c) The characteristic polynomial of A is

$$(2-t)(5-t) - (3-3i)(3+3i) = t^2 - 7t - 8 = (t-8)(t+1).$$

Hence, $-1, 8$ are all the eigenvalues of A. Note that for any scalars a, b,

$$3\begin{pmatrix} -2 & 1-i \\ 1+i & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -6 & 3-3i \\ 3+3i & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (A - 8I)\begin{pmatrix} a \\ b \end{pmatrix} = \vec{0}$$

if and only if $b = (1+i)a$. In particular, $u = (1, 1+i)$ is an eigenvector of A corresponding to eigenvalue 8.

$$\|u\| = \sqrt{1\overline{1} + (1+i)\overline{(1+i)}} = \sqrt{3}.$$

On the other hand, for any scalars a, b,

$$3\begin{pmatrix} 1 & 1-i \\ 1+i & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3 & 3-3i \\ 3+3i & 6 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = (A + I)\begin{pmatrix} a \\ b \end{pmatrix} = \vec{0}$$

if and only if $a = (i-1)b$. In particular, $v = (i-1, 1)$ is an eigenvector of A corresponding to eigenvalue -1.

$$\|v\| = \sqrt{(i-1)\overline{(i-1)} + 1\overline{1}} = \sqrt{3}.$$

Then

$$P = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & i-1 \\ i+1 & 1 \end{pmatrix}$$

is a unitary matrix and

$$D = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}$$

is a diagonal matrix such that $P^*AP = D$.

7 Q: Prove if T is a unitary operator on a finite-dimensional inner product space V, then T has a unitary square root.

Sol: Let β be the standard ordered basis and $A = [T]_\beta$. By Theorem 6.19 we have a unitary matrix Q and a diagonal matrix D s.t.

$$A = Q^*DQ.$$

Since A is unitary, we have $A^*A = Q^*D^*QQ^*DQ = Q^*D^*DQ = I$ which implies $D^*D = I$. By the fact that D is diagonal, denote

$$D = \begin{pmatrix} |d_1|e^{i\theta_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & |d_n|e^{i\theta_n}. \end{pmatrix}$$

Then we have $|d_j| = 1$. Let $U = Q^* \begin{pmatrix} \sqrt{|d_1|}e^{\frac{i\theta_1}{2}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{|d_n|}e^{\frac{i\theta_n}{2}} \end{pmatrix} Q$. We can varify U satiesfies our requirements.

10 Q: Let A be an $n \times n$ real symmetric or complex normal matrix. Prove

$$\operatorname{tr}(A) = \sum_{i=1}^{n} \lambda_i \qquad \operatorname{tr}(A^*A) = \sum_{i=1}^{n} |\lambda_i|^2,$$

where the λ_i's are the eigenvalues of A.

Sol: There are unitary matrix Q and diagonal matrix D s.t. $A = Q^*DQ$ and the diagonal elements of D are eigenvalues of A. Then we have

$$\operatorname{tr}(A) = \operatorname{tr}(Q^*DQ) = \operatorname{tr}(Q^*QD) = \operatorname{tr}(D) = \sum_{i=1}^{n} \lambda_i.$$

$$\operatorname{tr}(A^*A) = \operatorname{tr}(Q^*D^*QQ^*DQ) = \operatorname{tr}(D^*D) = \operatorname{tr}(H) = \sum_{i=1}^{n} |\lambda_i|^2.$$

Sec. 6.6

6 Q: Let T be a normal operator on a finite-dimensional inner product space. Prove that if T is a projection, then T is also an orthogonal projection.

Sol: Let V be the domain of the operator T. Fix $u \in \mathsf{N}(T)$ and $w \in \mathsf{R}(T)$. We claim that $\langle u, v\rangle = 0$. If either u or w is the zero vector, then we are done. Now suppose $u \neq \vec{0}$ and $w \neq \vec{0}$. As $T(u) = \vec{0}$ and $T(w) = w$, u is indeed an eigenvector of T corresponding to the eigenvalue 0, while w is an eigenvector of T corresponding to the eigenvalue 1. By Theorem 6.15, $\langle u, w\rangle = 0$. Therefore, $\mathsf{N}(T)$ and $\mathsf{R}(T)$ are orthogonal, whence T is an orthogonal projection.

7 Q: Let T be a normal operator on a finite-dimensional complex inner product space V. Use the spectral decomposition $\lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k$ of T to prove the following results.

(a) If g is a polynomial, then

$$g(T) = \sum_{i=1}^{k} g(\lambda_i) T_i.$$

(b) If $T^n = T_0$ for some n, then $T = T_0$.

(c) Let U be a linear operator on V. Then U commutes with T if and only if U commutes with each T_i.

(d) There exists a normal operator U on V such that $U^2 = T$.

(e) T is invertible if and only if $\lambda_i \neq 0$ for $1 \leq i \leq k$.

(f) T is a projection if and only if every eigenvalue of T is 1 or 0.

(g) $T = -T^*$ if and only if every λ_i is an imaginary number.

Sol: (a) Note that $T^0 = I = \sum_{i=1}^{k} T_i$. $\forall j \in \mathbb{Z}^+$,

$$T^j = \sum_{i_1=1}^{k} \cdots \sum_{i_j=1}^{k} \lambda_{i_1} \cdots \lambda_{i_j} T_{i_1} \cdots T_{i_j} = \sum_{i_1=1}^{k} \cdots \sum_{i_j=1}^{k} \lambda_{i_1} \cdots \lambda_{i_j} \delta_{i_1 i_2} \delta_{i_1 i_3} \cdots \delta_{i_1 i_j} T_{i_1}$$
$$= \sum_{i=1}^{k} \lambda_i^j T_i.$$

Write $g(t) = a_n t^n + \cdots + a_1 t + a_0$, where $a_0, ..., a_n \in \mathbb{C}$. Then

$$g(T) = a_n T^n + \cdots + a_1 T + a_0 I = a_n \sum_{i=1}^{k} \lambda_i^n T_i + \cdots a_1 \sum_{i=1}^{k} \lambda_i T_i + a_0 \sum_{i=1}^{k} T_i$$
$$= \sum_{i=1}^{k} (a_n \lambda_i^n + \cdots a_1 \lambda_i + a_0) T_i = \sum_{i=1}^{k} g(\lambda_i) T_i.$$

(b) Suppose $T^n = T_0$ for some n. Then $\sum_{i=1}^{k} \lambda_i^n T_i = T_0$. It implies that $\lambda_1^n = \cdots = \lambda_k^n = 0$, whence $\lambda_1 = \cdots = \lambda_k = 0$. Therefore, $T = T_0$.

(c) ($\Rightarrow$) Since T, U commute, a T-invariant subspace of V is also U-invariant. Fix $v \in V$. $\forall i \in \{1, ..., k\}$, we have

$$T_i U(v) + (T - (\lambda_i - 1)T_i)U(v) = TU(v) = UT(v) = UT_i(v) + U(T - (\lambda_i - 1)T_i)(v)$$

and therefore $T_i U(v) = U T_i(v)$.

($\Leftarrow$) We have

$$UT = \lambda_1 U T_1 + \cdots + \lambda_k U T_k = \lambda_1 T_1 U + \cdots + \lambda_k T_k U = TU.$$

(d) $\forall i \in \{1, ..., k\}$, choose $\mu_i \in \mathbb{C}$ such that $\mu_i^2 = \lambda_i$. Define $U = \mu_1 T_1 + \cdots + \mu_k T_k$. By Gram-Schmidt Orthogonalization Process and Theorem 6.16, U is normal. Using the result of (a), $U^2 = \mu_1^2 T_1 + \cdots + \mu_k^2 T_k = \lambda_1 T_1 + \cdots + \lambda_k T_k = T$.

(e) ($\Rightarrow$) In particular, $\mathsf{N}(T) = \{\vec{0}\}$. Then 0 is not an eigenvalue of T, whence $\lambda_i \neq 0$ for $1 \leq i \leq k$.
($\Leftarrow$) It means that 0 is not an eigenvalue of T. So if $v \in \mathsf{N}(T)$, then $T(v) = \vec{0} = 0 \cdot v$, forcing that $v = \vec{0}$. T is then one-to-one. As V is finite-dimensional, T is also onto. Then T is invertible.

(f) ($\Rightarrow$) Suppose $\lambda \in \mathbb{C}$ is an eigenvalue of T. Then $\exists v \in V$ such that $v \neq \vec{0}$ and $T(v) = \lambda v$. As T is a projection, $\lambda v = T(v) = T^2(v) = \lambda^2 v$, whence $\lambda(\lambda - 1)v = \vec{0}$. As $v \neq \vec{0}$, $\lambda(\lambda - 1) = 0$, whence either $\lambda = 1$ or $\lambda = 0$.
($\Leftarrow$) Case (1): Suppose 1 is an eigenvalue of T. Then without loss of generality we can assume $\lambda_1 = 1$ and $\lambda_i = 0$ for any $1 < i \leq k$. Then $T = T_1$ is a projection.
Case (2): Suppose 1 is not eigenvalue of T. Then without loss of generality we can assume $\lambda_i = 0$ for any $1 \leq i \leq k$ and hence T is the zero transformation, which is a projection as well.

(g) ($\Rightarrow$) Fix $i \in \{1, ..., k\}$. Fix v_i with $v_i \neq \vec{0}$ and $T(v_i) = \lambda_i v_i$. Then $T^*(v_i) = \overline{\lambda_i} v_i$. We have $\lambda_i v_i = T(v_i) = -T^*(v_i) = -\overline{\lambda_i} v_i$. But $v_i \neq \vec{0}$. Thus, $\lambda_i = -\overline{\lambda_i}$. It means that λ_i is an imaginary number.
($\Leftarrow$) Fix $v \in V$. Then $\exists v_1, ..., v_k \in V$ such that $T(v_i) = \lambda_i v_i$ $\forall i \in \{1, ..., k\}$ and $v = v_1 + \cdots + v_k$. We have

$$-T^*(v) = -T^*(v_1) - \cdots - T^*(v_k) = -\overline{\lambda_1} v_1 - \cdots - \overline{\lambda_k} v_k = \lambda_1 v_1 + \cdots + \lambda_k v_k = T(v).$$

Therefore, $T = -T^*$.

Optional Part

Sec. 6.4

1 Q: Label the following statements as true or false. Assume that the underlying inner product spaces are finite-dimensional.

(a) Every self-adjoint operator is normal.
(b) Operators and their adjoints have the same eigenvectors.
(c) If T is an operator on an inner product space V, then T is normal if and only if $[T]_\beta$ is normal, where β is any ordered basis for V.
(d) A real or complex matrix A is normal if and only if L_A is normal.
(e) The eigenvalues of a self-adjoint operator must all be real.
(f) The identity and zero operators are self-adjoint.
(g) Every normal operator is diagonalizable.
(h) Every self-adjoint operator is diagonalizable.

Sol: (a) True.
(b) False.

(c) False.
(d) True.
(e) True.
(f) True.
(g) False.
(h) True.

8 Q: Let T be a normal operator on a finite-dimensional complex inner product space V, and let W be a subspace of V. Prove that if W is T-invariant, then W is also T^*-invariant.

Sol: If W is the zero subspace of V, then clearly W is T^*-invariant. Now assume W is not the zero subspace of V. By Theorem 6.16, there exists an orthonormal basis for V consisting of eigenvectors of T, implying that T is diagonalizable. By Exercise 24 in Sec. 5.4, as W is T-invariant, T_W is diagonalizable. Hence,

$$W = \mathsf{E}_{\lambda_1} \oplus \cdots \oplus \mathsf{E}_{\lambda_k},$$

where $\lambda_1, ..., \lambda_k$ are all the distinct eigenvalues of T_W and E_{λ_i} is the eigenspace of T_W corresponding to eigenvalue λ_i $\forall i \in \{1, ..., k\}$. Fix $w \in W$. $\exists$ vectors $v_1 \in \mathsf{E}_{\lambda_1}$, ..., $v_k \in \mathsf{E}_{\lambda_k}$ such that $w = v_1 + \cdots + v_k$. $\forall i \in \{1, ..., k\}$, as $T(v_i) = T_W(v_i) = \lambda_i v_i$, by Theorem 6.15, $T^*(v_i) = \overline{\lambda}_i v_i \in \mathsf{E}_{\lambda_i}$. Then $T^*(w) \in W$. Therefore, W is T^*-invariant.

10 Q: Let T be a self-adjoint operator on a finite-dimensional inner product space V. Prove that for all $x \in V$

$$\|T(x) \pm ix\|^2 = \|T(x)\|^2 + \|x\|^2.$$

Deduce that $T - iI$ is invertible and that $[(T - iI)^{-1}]^* = (T + iI)^{-1}$.

Sol: Fix $x \in V$. $\langle T(x), ix\rangle = \langle x, T(ix)\rangle = \langle x, iT(x)\rangle = \langle \bar{i}x, T(x)\rangle = -\langle ix, T(x)\rangle = -\langle T(x), ix\rangle$. Therefore, $\langle T(x), ix\rangle = 0$. We have

$$\|T(x) \pm ix\|^2 = \|T(x)\|^2 + \|\pm ix\|^2 \pm 2\langle T(x), ix\rangle = \|T(x)\|^2 + \|x\|^2.$$

Suppose $x \in \ker(T - iI)$. Then $\|x\|^2 \leq \|x\|^2 + \|T(x)\|^2 = \|T(x) - ix\|^2 = 0$. It forces that $\|x| = 0$ and thus $x = \vec{0}$. Therefore, $T - iI$ is one-to-one. As V is of finite dimension, $T - iI$ is also onto. Therefore, $T - iI$ is invertible.
Since T is self-adjoint, $\forall u, v \in V$,

$$\langle u, (T + iI)^*(v)\rangle = \langle (T + iI)(u), v\rangle = \langle u, (T - iI)(v)\rangle.$$

Thus, $(T + iI)^* = T - iI$. Then $(T - iI)^{-1}(T + iI)^* = I$.

$$(T + iI)[(T - iI)^{-1}]^* = ((T + iI)^*)^*[(T - iI)^{-1}]^* = [(T - iI)^{-1}(T + iI)^*]^* = I^* = I.$$

Therefore, $[(T - iI)^{-1}]^* = (T + iI)^{-1}$.

12 Q: Let T be a normal operator on a finite-dimensional real inner product space V whose characteristic polynomial splits. Prove that V has an orthonormal basis of eigenvectors of T. Hence prove that T is self-adjoint.

Sol: By Schur's Theorem (Theorem 6.14), there exists an orthonormal basis $\beta = (e_1, ..., e_n)$ for V such that the matrix $A = [T]_\beta$ is upper triangular. Then e_1 is an eigenvector of T corresponding to eigenvalue A_{11}. Assume $k \in \{2, ..., n\}$ and $e_1, ..., e_{k-1}$ are eigenvectors of T. $\forall i \in \{1, ..., k-1\}$, let λ_i be the eigenvalue of T corresponding to the eigenvector e_i of T. Since A is upper triangular,

$$T(e_k) = A_{1k}v_1 + \cdots + A_{kk}e_k.$$

Then $\forall i \in \{1, ..., k-1\}$,

$$A_{ik} = \langle T(e_k), e_i\rangle = \langle e_k, T^*(e_i)\rangle = \langle e_k, \overline{\lambda}_i e_i\rangle = \lambda_i \langle e_k, e_i\rangle = \vec{0}.$$

We have $T(e_k) = A_{kk}e_k$. Thus, e_k is an eigenvector of T. By mathematical induction, β is an orthonormal basis of eigenvectors of T. Then by Theorem 6.17, T is self-adjoint.

14 Q: *Simultaneous Diagonalization.* Let V be a finite-dimensional real inner product space, and let U and T be self-adjoint linear operators on V such that $UT = TU$. Prove that there exists an orthonormal basis for V consisting of vectors that are eigenvectors of both U and T.

Sol: Let $\lambda_1, ..., \lambda_k$ be all the distinct eigenvalues of T. $\forall i \in \{1, ..., k\}$, let E_{λ_i} be the eigenspace of T corresponding to the eigenvalue λ_i. By Theorem 6.17, we have an orthogonal decomposition

$$V = \mathsf{E}_{\lambda_1} \oplus \cdots \oplus \mathsf{E}_{\lambda_k}.$$

Fix $i \in \{1, ..., k\}$. Since $TU = UT$, E_{λ_i} is U-invariant. Then by Exercise 7 in Sec. 6.4, $U_{\mathsf{E}_{\lambda_i}}$ is self-adjoint because U is self-adjoint. By Theorem 6.17, $\exists$ orthonormal basis $\{v_{ii}, ..., v_{in_i}\}$ of $U_{\mathsf{E}_{\lambda_i}}$ for E_{λ_i} such that $v_{ii}, ..., v_{in_i}$ are eigenvectors of $U_{\mathsf{E}_{\lambda_i}}$. Then

$$\beta = \{v_{11}, ..., v_{1n_1}, ..., v_{k1}, ..., v_{kn_k}\}$$

is an orthonormal basis for V such that $\forall i \in \{1, ..., k\}$, $\forall j \in \{1, ..., n_i\}$, v_{ij} is an eigenvector of both U and T.

Sec. 6.5

1 Sol: (a) True.
(b) False.
(c) False.
(d) True.
(e) False.
(f) Ture.
(g) False.
(h) False.
(i) False.

9 Sol: If V is one dimensional, then correct since $||Uy|| = ||Ucx|| = c||x|| = ||y||$. If $\dim V > 1$. Then false since if β_1 is one of the orthogonal basis, we can define U as $Ux = \beta_1$ for all $x \in V$ in the orthogonal basis. If $\beta_2 \neq \beta_1$ is one of the orthogonal basis, we have $\|\beta_1 - \beta_2\| \neq 0 = \|U\beta_1 - U\beta_2\|$.

12 Sol: There are unitary matrix Q and diagonal matrix D s.t. $A = Q^*DQ$ and the diagonal elements of D are eigenvalues $\{\lambda_i\}_{i=1}^n$ of A. Then $\det(A) = \det(Q^*DQ) = \det(D) = \Pi_{i=1}^n \lambda_i$.

15 Sol: (a) Since U is W-invariant, we have $U(W) \subseteq W$. It then suffices to show that $W \subseteq U(W)$.

Consider $U_W : W \to W$, the restriction of U on W. Then U_W is linear. As U is unitary, $\|U(v)\| = \|v\|$ for all $v \in V$. In particular, $\|U_W(w)\| = \|U(w)\| = \|w\|$ for all $w \in$. So U_W is one-to-one. As W is finite dimensional, U_W is then onto, and so $W \subseteq U_W(W) = U(W)$.

Hence we have $U(W) = W$.

(b) Let $v \in W^\perp$. Then $\langle v, w\rangle = 0$ for all $w \in W$. Let $w \in W$. By the previous question there exists $w' \in W = U(W)$ such that $w = Uw'$. Then $\langle Uv, w\rangle = \langle v, U^*Uw'\rangle = \langle v, w'\rangle = 0$. As $w' \in W$ is arbitrary, $Uv \in W^\perp$.

As $v \in W^\perp$ is arbitrary, $W^\perp$ is U-invariant.

16 Sol: Let $V = \ell^2(\mathbb{R})$, the space of real square-summable sequences, equipped with the inner product $\langle (a_k), (b_k)\rangle = \sum a_k b_k$. Let $W = \{(a_i) \in V : a_{2k-1} = 0, \forall k \in \mathbb{Z}^+\}$. Then W is a subspace of V. Let $U : V \to V$ be defined by $U((a_k)) = (b_k)$ where $b_{2k-1} = a_{2k+1}$, $b_2 = a_1$, $b_{2k+2} = a_{2k}$ for $k \in \mathbb{Z}^+$. It is easy to verify that U is well-defined linear and unitary. Also, $U(W) \subseteq W$. Let $e = (e_k)$ be the sequence with entries $e_k = \delta_{1,k}$ where the first entry is 1 and all other entries are 0. Then $e \in W^\perp$ but $0 \neq U(e) \in W$ and so $U(e) \notin W^\perp$. In particular, $W^\perp$ is not U-invariant.

17 Sol: We prove by induction on the dimension n. When $n = 1$, the argument holds. Assume the statement holds when $n = k$, then when $n = k + 1$, let A be a unitary and upper triangular matrix and

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ & & a_{nn}. \end{pmatrix}$$

Then

$$A^*A = \begin{pmatrix} \overline{a_{11}} & & \\ \vdots & \ddots & \\ \overline{a_{1n}} & \cdots & \overline{a_{nn}} \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ & & a_{nn}. \end{pmatrix}$$

Since $A^*A = I$, the first row of

$$A^*A = \begin{pmatrix} |a_{11}^2| & \overline{a_{11}}a_{12} & \cdots & \overline{a_{11}}a_{1n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}.$$

Then we can see all $a_{1i} = 0$ except for $i = 1$. Then

$$A = \begin{pmatrix} a_{11} & \cdots & 0 \\ & \ddots & \vdots \\ & & a_{nn}. \end{pmatrix} = \begin{pmatrix} a_{11} & 0 \\ 0 & A', \end{pmatrix}$$

where A' is a $k \times k$ unitary and upper triangular matrix, then A' is diagonal, hence A is diagonal.

Sec. 6.6

1 Q: Label the following statements as true or false. Assume that the underlying inner product spaces are finite-dimensional.

(a) All projections are self-adjoint.

(b) An orthogonal projection is uniquely determined by its range.

(c) Every self-adjoint operator is a linear combination of orthogonal projections.

(d) If T is a projection on W, then $T(x)$ is the vector in W that is closest to x.

(e) Every orthogonal projection is a unitary operator.

Sol: (a) False.

(b) True.

(c) True.

(d) False.

(e) False.

4 Q: Let W be a finite-dimensional subspace of an inner product space V. Show that if T is the orthogonal projection of V on W, then $I - T$ is the orthogonal projection of V on $W^{\perp}$.

Sol: Fix $v \in V$. Then $\exists$ unique $w \in W$ and unique $u \in W^{\perp}$ such that $v = w + u$. As T is the orthogonal projection of V on W, $w = T(v)$ and thus $u = v - w = (I - T)(v)$. Therefore, $I - T$ is a projection of V on $W^{\perp}$ along $W = (W^{\perp})^{\perp}$, which implies that $I - T$ is the orthogonal projection of V on $W^{\perp}$.

10 Q: *Simultaneous diagonalization.* Let U and T be normal operators on a finite-dimensional complex inner product space V such that $TU = UT$. Prove that there exists an orthonormal basis for V consisting of vectors that are eigenvectors of both T and U.

Sol: Let $\lambda_1, ..., \lambda_k$ be all the distinct eigenvalues of T. $\forall i \in \{1, ..., k\}$, let E_{λ_i} be the eigenspace of T corresponding to the eigenvalue λ_i. By Theorem 6.16, we have an orthogonal decomposition

$$V = \mathsf{E}_{\lambda_1} \oplus \cdots \oplus \mathsf{E}_{\lambda_k}.$$

Fix $i \in \{1, ..., k\}$. Since $TU = UT$, E_{λ_i} is U-invariant. Note that E_{λ_i} is the eigenspace of T^* corresponding to eigenvalue $\overline{\lambda}_i$. We also have $T^*U^* = (UT)^* = (TU)^* = U^*T^*$ and thus E_{λ_i} is also U^*-invariant. Then by Exercise 7 in Sec. 6.4, $U_{\mathsf{E}_{\lambda_i}}$ is normal because U is normal. By Theorem 6.16, $\exists$ orthonormal basis $\{v_{ii}, ..., v_{in_i}\}$ of $U_{\mathsf{E}_{\lambda_i}}$ for E_{λ_i} such that $v_{ii}, ..., v_{in_i}$ are eigenvectors of $U_{\mathsf{E}_{\lambda_i}}$. Then

$$\beta = \{v_{11}, ..., v_{1n_1}, ..., v_{k1}, ..., v_{kn_k}\}$$

is an orthonormal basis for V such that $\forall i \in \{1, ..., k\}$, $\forall j \in \{1, ..., n_i\}$, v_{ij} is an eigenvector of both U and T.