Topic#13
Inner product space

V: v.s. over $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, with "$+$" & "$\cdot$"

Goal: From a vector space V, introduce an inner product over V to discuss the length, distance, orthogonality of vectors in V, finanlly get an inner product space V.

$\rightsquigarrow$ more geometric applications

Eg.1 Projection of a vector to a plane.

Eg.2 $\beta = \{v_1, \cdots, v_n\}$ is an o.b. for V. Through Gram-Schmidt process, we get $\beta' = \{v_1', \cdots, v_n'\}$ is an orthogonal o.b. for V.

Eg.3 $A \in M_{n \times n}(\mathbb{F})$ diagonalizable if and only if $\exists$ an o.b. β for $\mathbb{F}^n$. consisting entirely of e-vectors of A. In what situation, can β be orthogonal?

$\mathbb{F} = \mathbb{C}$: A is a normal matrix.

$\mathbb{F} = \mathbb{R}$: A is a symmetric matrix.

Def. V: v.s. over $\mathbb{F}$ ($=\mathbb{C}$ or $\mathbb{R}$) An **inner product** on V is a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}, \quad (x, y) \mapsto \langle x, y \rangle$$

such that

(a) $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle, \forall x, y, z \in V$.

(b) $\langle cx, y \rangle = c\langle x, y \rangle, \forall x, y, \in V, \forall c \in \mathbb{F}$.

(c) $\overline{\langle x, y \rangle} = \langle y, x \rangle, \forall x, y \in V$.

(d) If $x \neq 0_v$, then $\langle x, x \rangle$ is real and $\langle x, x \rangle > 0$.

Complx conjugate: $\overline{a + bi} = a - bi, a, b \in \mathbb{R}$

Note:

1°. $\langle \cdot, \cdot \rangle$ is linear in the first component (Exercise):

$$\langle \sum_{i=1}^{m} a_i x_i, y \rangle = \sum_{i=1}^{m} a_i \langle x_i, y \rangle.$$

2°. If $\mathbb{F} = \mathbb{R}$. then condition (c) reduces to $\langle x, y \rangle = \langle y, x \rangle$.

Quick properties of inner products:

Theorem. Let V be an i.p.s. with $\langle \cdot, \cdot \rangle$. Then,
(1) $\langle x, y+z \rangle = \langle x, y \rangle + \langle y, z \rangle, \forall x, y, z \in V$.
(2) $\langle x, cy \rangle = \bar{c}\langle x, y \rangle, \forall x, y, \in V, \forall c \in \mathbb{F}$.
(3) $\langle x, 0_v \rangle = \langle 0_v, x \rangle = 0, \forall x \in V$.
(4) $\langle x, x \rangle = 0$ iff $x = 0$.
(5) If $\langle x, y \rangle = \langle x, z \rangle$ for any $x \in V$, then $y = z$. Particularly, if $\langle x, y \rangle = 0$ for any $x \in V$, then $y = 0_v$.

Proof:

(1): $\overline{\langle x, y+z\rangle} \overset{(c)}{=} \langle y+z, x\rangle \overset{(a)}{=} \langle y, x\rangle + \langle z, x\rangle \overset{(c)}{=} \overline{\langle x, y\rangle} + \overline{\langle x, z\rangle}$
$= \overline{\langle x, y\rangle + \langle x, z\rangle}$ □

(2): $\overline{\langle x, cy\rangle} \overset{(c)}{=} \langle cy, x\rangle \overset{(b)}{=} c\langle y, x\rangle = \cdots = \overline{\overline{c}\langle x, y\rangle}$ □

(3): $\langle x, 0_v\rangle = \langle x, 0\cdot 0_v\rangle = \overline{0}\langle x, 0_v\rangle = 0$ □

(4): $\langle x, x\rangle = 0$ iff $x = 0_v$:
$\Leftarrow$ obvious by (3)
$\Rightarrow$ otherwise, $x \neq 0_v$, then by (d), $\langle x, x\rangle > 0$ contradiction
then $x = 0_v$ □

(5): Easy proof in the simple situation: $\langle x, y\rangle = 0\,\forall x \in V \Rightarrow y = 0$
Generally, $\langle x, y\rangle = \langle x, z\rangle$, $\forall x \in V$, then
$0 = \langle x, y\rangle - \langle x, z\rangle = \langle x, y-z\rangle, \forall x \in V$. Apply the simple situation, $y - z = 0_v$, $\therefore y = z$. □

Note:

1°. (a) & (b) mean:

$$\langle x, \sum_{i=1}^{m} b_i y_i \rangle = \sum_{i=1}^{m} \bar{b}_i \langle x, y_i \rangle$$

i.p is conjugate linear in the 2^{nd} component.

Therefore

$$\langle \sum_{i=1}^{m} a_i x_i, \sum_{i=1}^{n} b_i y_i \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i \bar{b}_j \langle x_i, y_j \rangle.$$

2°. (e) means:

Let $x \in V$. If $\langle x, y \rangle = 0$ for all $y \in V$, then $x = 0$.

It is a useful characterization of zero vector.

Def.: A v.s. V over $\mathbb{F}$ equipped with an inner product is called an **inner product space**.

Remarks:

1°. V is a complex i.p.s. if $\mathbb{F} = \mathbb{C}$; V is a real i.p.s. if $\mathbb{F} = \mathbb{R}$.

2°. Let V be an i.p.s. with the i.p. $\langle \cdot, \cdot \rangle$.
Let W be a subspace of V.
Then W equipped with the same $\langle \cdot, \cdot \rangle$ is also an i.p.s.

3°. A v.s. V can be equipped with two different inner products and then this yields two different inner product spaces. For instance, for $V = P(\mathbb{R})$, we may let

$$\langle f, g \rangle_1 = \int_0^1 f(t)g(t)dt$$

or

$$\langle f, g \rangle_2 = \int_{-1}^1 f(t)g(t)dt.$$

These two are **distinct** inner products (why?).

Examples of inner products.

e.g. 1: $V = \mathbb{F}^n$. (scalar field $= \mathbb{C}$ or $\mathbb{R}$)

$x = (x_1, \cdots, x_n) \in \mathbb{F}^n$
$y = (y_1, \cdots, y_n) \in \mathbb{F}^n$

$$\langle x, y \rangle \overset{def}{=} \sum_{i=1}^{n} x_i \bar{y}_i.$$

Verify: $\langle \cdot, \cdot \rangle$ is an i.p. over $\mathbb{F}^n$. (Exercise)

Note:

1°. We call it the standard i.p. over $\mathbb{F}^n$.

2°. If $\mathbb{F} = \mathbb{R}$, then

$$\langle x, y \rangle \overset{def}{=} \sum_{i=1}^{n} x_i y_i \text{ (dot product } x \cdot y \text{ in } \mathbb{R}^n).$$

e.g. 2: $V = C([0,1])$ (set of real-valued continuous f'ns on $[0,1]$) with $\mathbb{F} = \mathbb{R}$. Let

$$\langle \cdot, \cdot \rangle : (f, g) \in V \times V \mapsto \langle f, g \rangle \overset{def}{=} \int_0^1 f(t)g(t)dt \in \mathbb{R}.$$

Verify: $\langle \cdot, \cdot \rangle$ is an inner product on $V = C([0,1])$.

- $\langle \cdot, \cdot \rangle$ is well-defined (obvious).
- Show (a), (b), (c), (d):

$$\left.\begin{array}{l}(a)\\(b)\\(c)\end{array}\right\} \textit{obvious}$$

(d): Let $0 \neq f \in C([0,1])$, then

$$\langle \cdot, \cdot \rangle = \int_0^t f^2(t)dt > 0.$$

($\because$ $f^2 > 0$ on some subinterval of $[0,1]$) $\square$

e.g. 3:
An example to be used later: Let

$$H \stackrel{def}{=} \{\text{continuous complex-valued functions on } [0, 2\pi]\},$$

then H is a v.s. over $\mathbb{F}$. Let

$$\langle f, g \rangle \stackrel{def}{=} \frac{1}{2\pi} \int_0^{2\pi} f(t)\overline{g(t)}dt, \quad f, g \in H.$$

Show: H with $\langle \cdot, \cdot \rangle$ as above is an i.p.s. (Exercise)

Note: $f \in H \Leftrightarrow f = f_1 + if_2$ for some $f_1, f_2 \in C([0, 2\pi])$.

e.g. 4: $V = M_{n\times n}(\mathbb{F})$ ($\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$). Let

$$\begin{aligned} \langle\cdot,\cdot\rangle\colon\ & V \times V \to \mathbb{F} \\ & (A, B) \mapsto \langle A, B\rangle \overset{def}{=} tr(B^* A). \end{aligned}$$

Verify: $\langle\cdot,\cdot\rangle$ is an i.p. on V.

Notation: For $A = [A_{ij}]_{n\times n} \in M_{n\times n}(\mathbb{F})$,

$$\mathbb{F} \ni tr(A) \overset{def}{=} \textstyle\sum_{i=1}^n A_{ii} \text{ : trace of } A$$

$$M_{n\times n}(\mathbb{F}) \ni A^* \overset{def}{=} \bar{A}^t, i.e.\ (A^*)_{ij} = \bar{A}_{ji}\text{: conjugate transpose or adjoint of } A.$$

Proof.

1°. $\langle\cdot,\cdot\rangle$ is well-defined. (obvious)

2^o. Show (a), (b), (c), (d) in the definition:

Let $A, B, C \in M_{n\times n}(\mathbb{F}), c \in \mathbb{F}$.

(a)

$$\langle A+B, C\rangle = tr(C^*(A+B)) = tr(C^*A + c^*B)$$
$$= tr(C^*A) + tr(C^*B) = \langle A, C\rangle + \langle B, C\rangle.$$

(b) $\langle cA, B\rangle = tr(B^*(cA)) = c\,tr(B^*A) = c\langle A, B\rangle.$

(c)

$$\overline{\langle A, B\rangle} = \overline{tr(B^*A)} = \overline{tr(B^*A)^t} = \overline{tr(\bar{B}^t A)^t} = \overline{tr(A^t\bar{B})}$$
$$= tr(\bar{A}^t B) = tr(A^*B) = \langle B, A\rangle.$$

(d)

$$\langle A, A\rangle = tr(A^*A) = \textstyle\sum_{i=1}^n (A^*A)_{ii} = \sum_{i=1}^n \sum_{j=1}^n (A^*)_{ij} A_{ji}$$
$$= \textstyle\sum_{i=1}^n \sum_{i=1}^n \bar{A}_{ji} A_{ji} = \sum_{i=1}^n \sum_{j=1}^n |A_{ji}|^2.$$

Therefore,

$$A \neq 0 \Rightarrow \langle A, A\rangle > 0.$$

(Equivalently, $\langle A, A\rangle = 0 \Rightarrow A = 0$ (zero matrix))

Note: $\langle A, B\rangle = tr(B^*A)$ called the Frobenius i.p. □

Def. Let V be an i.p.s. with the i.p. $\langle \cdot, \cdot \rangle$. For $x \in V$,

$$\|x\| \stackrel{def}{=} \sqrt{\langle x, x \rangle}$$

is called the **norm** of $x \in V$, which is an i.p.s with $\langle \cdot, \cdot \rangle$.

e.g. $V = \mathbb{F}^n \ni x = (x_1, \cdots, x_n)$:

$$\|x\| \stackrel{def}{=} \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^{n} x_i \bar{x}_i} = (\sum_{i=1}^{n} |x_i|^2)^{1/2}.$$

$\uparrow$
Euclidean *length* of $x \in \mathbb{F}^n$.

Theorem. Let V be an i.p.s. over $\mathbb{F}$ with $\langle\cdot,\cdot\rangle$. Then,
(1) $\|x\| \geq 0, \forall x \in V$. And, $\|x\| = 0$ iff $x = 0_v$.
(2) $\|cx\| = |c| \cdot \|x\|, \forall x \in V, \forall c \in \mathbb{F}$.
(3) $\|x+y\| \leq \|x\| + \|y\|, \forall x, y \in V$. (Triangle Inequality)

Pf.
Proof of (1) & (2): based on def of $\|\cdot\|$: $\|x\| = \sqrt{\langle x, x\rangle}$.

To show Thm(3), we need to show two lemmas.

Lemma1 (Pythogorean Thm):

If $\langle x, y\rangle = 0$, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

Proof: $\|x+y\|^2 \stackrel{def}{=} \langle x+y, x+y\rangle$

$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle = \langle x, x\rangle + \langle y, y\rangle = \|x\|^2 + \|y\|^2$

□

Lemma2 (Cauchy-Schwarz inequality)

$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|, \forall x, y \in V.$

Proof: Let $x, y \in V$. Ture if $x = 0_v$ or $y = 0_v$. Assume: $x \neq 0_v$ and $y \neq 0_v$.

Then take $z \overset{def}{=} y - \frac{\langle y,x\rangle}{\langle x,x\rangle}x$,

then $\langle z, x\rangle = \langle y - \frac{\langle y,x\rangle}{\langle x,x\rangle}x, x\rangle = \langle y, x\rangle - \frac{\langle y,x\rangle}{\langle x,x\rangle}\langle x, x\rangle = 0$

Apply lemma 1 above to $y = z + \frac{\langle y,x\rangle}{\langle x,x\rangle}x$:

$$\|y\|^2 = \left\|z + \frac{\langle y,x\rangle}{\langle x,x\rangle}x\right\|^2 \overset{lemma1}{=} \|z\|^2 + \left\|\frac{\langle y,x\rangle}{\langle x,x\rangle}x\right\|^2 = \|z\|^2 + \frac{|\langle y,x\rangle|^2}{\|x\|^2}$$

$\therefore \|y\|^2 \geq \frac{|\langle y,x\rangle|^2}{\|x\|^2}$

$\therefore |\langle x, y\rangle|^2 \leq \|x\|\|y\|.$

□

Now we can show Thm(3), the triangle inequality.

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ &= \langle x,x\rangle + (\langle x,y\rangle + \overline{\langle x,y\rangle}) + \langle y,y\rangle \\ &= \|x\|^2 + 2\mathrm{Re}\langle x,y\rangle + \|y\|^2 \\ &\le \|x\|^2 + 2|\langle x,y\rangle| + \|y\|^2 \ (-|z| \le \mathrm{Re} z \le |z|) \\ &\le \|x\|^2 + 2\|x\|\cdot\|y\| + \|y\|^2 \ (\text{c.-s.}) \\ &= (\|x\| + \|y\|)^2.\end{aligned}$$

So, $\|x+y\| \le \|x\| + \|y\|$.

In fact, we can also construct a quadratic equation $0 \le f(t) \stackrel{def}{=} \|x+ty\|^2 = \cdots = a + bt + ct^2$ where $c \neq 0$

$\therefore b^2 - 4ac \le 0$. It implies C.S. inequality.

Examples for applications of C.S. Inequality & Triangle Inequality

- $V = \mathbb{F}^n$ with the s.i.p.:

$$|\sum_{i=1}^{n} x_i y_i| \leq (\sum_{i=1}^{n} |x_i|)^{1/2} (\sum_{i=1}^{n} |y_i|^2)^{1/2}.$$

$$(\sum_{i=1}^{n} |x_i + y_i|^2)^{1/2} \leq (\sum_{i=1}^{n} |x_i|^2)^{1/2} + (\sum_{i=1}^{n} |y_i|^2)^{1/2}$$

- $V = C([a,b])$ with $\langle f, g \rangle = \int_a^b f(t) g(t) dt$:

$$\left| \int_a^b f(t) g(t) dt \right| \leq \left(\int_a^b |f(t)|^2 dt \right)^{1/2} \left(\int_a^b |g(t)|^2 dt \right)^{1/2},$$

$$\left(\left| \int_a^b |f(t) + g(t)|^2 dt \right| \right)^{1/2} \leq \left(\int_a^b |f(t)|^2 dt \right)^{1/2} + \left(\int_a^b |g(t)|^2 dt \right)^{1/2}.$$

$\square$