Topic#8
Invertibility & Isomorphism

Goal:

Let $T \in \mathcal{L}(V, W)$
with $n =$dim(V)$<\infty$ and $m =$dim(W)$<\infty$
and α o.b. for V, β o.b. for W,

then, T is bijective if and only if $m = n$ and $[T]_\alpha^\beta$ is non-singular.

Note:
$A \in M_{n\times n}(F)$ is non-singular
$\iff$ $det(A) \neq 0$
$\iff$ A is invertible

Def. $T \in \mathcal{L}(V, W)$. T is **invertible** if there exists a function

$$U : W \to V$$

such that

$$TU = I_W \text{ and } UT = I_V.$$

Remark(1) T is invertible **iff** T is bijective.
Pf:

$\Rightarrow$ (a) T is onto. Indeed, let $y \in W$,

then, $T(U(y) = TU(y) = I_W(y) = y$

i.e. $\exists U(u) \in V$ s.t. $T(U(y)) = y$.

(b) $T : V \to W$ is one-to-one. Indeed, let $T(x) = T(y), x, y \in V$, then $U(T(x)) = U(T(y))$ i.e. $UT(x) = UT(y)$ i.e. $I_V(x) = I_V(y)$ i.e. $x = y$ $\square$

$\Leftarrow U \overset{def}{=} T^{-1}$

Remark(2) If T is invertible then U is unique, given $U = T^{-1}$.

Basic facts:

(1) If $T : V \to W$ is invertible then $T^{-1} : W \to V$ is invertible and $(T^{-1})^{-1} = T$.

(2) If $T : V \to W$ and $U : W \to Z$ are invertible, then

$$UT : V \to Z \text{ is invertible and } (UT)^{-1} = T^{-1}U^{-1}.$$

e.g.: Let $A \in M_{n\times n}(\mathbb{F})$, and

$x \in \mathbb{F}^n \mapsto L_A(x) = Ax \in \mathbb{F}^n$

Then,

$$L_A \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^n) \text{ is invertible } \textbf{iff } A \in M_{n\times n}(\mathbb{F}) \text{ is invertible.}$$

In this case,

$$(L_A)^{-1} = L_{A^{-1}}.$$

$\square$

Thm. If $T \in \mathcal{L}(V, W)$ is invertible

then $T^{-1} : W \to V$ is linear, so $T^(-1) : W \to V$ is linear.

Proof. Let $y_1, y_2 \in W, a \in \mathbb{F}$.
$\because$ T is invertible $\therefore$ T is bijective, T^{-1} exists uniquely.

$\therefore \exists! x_1 = T^{-1}(y_1), x_2 = T^{-1}(y_2) \in V$, s.t.

$$y_1 = T(x_1), y_2 = T(x_2).$$

Then,

$$\begin{aligned} & T^{-1}(a_1 y_1 + a_2 y_2) \\ & = T^{-1}(a_1 T(x_1) + a_2 T(x_2)) \\ & = T^{-1}(T(a_1 x_1 + a_2 x_2)) \\ & = a_1 x_1 + a_2 x_2 \\ & = a_1 T^{-1}(y_1) + a_2 T^{-1}(y_2) \end{aligned}$$

$\square$

Lemma. Let $T \in \mathcal{L}(V, W)$ be invertible. Then

$$\dim(V) < \infty \textbf{ iff } \dim(W) < \infty.$$

In this case, $\dim(V) = \dim(W) < \infty$.

Proof. "$\Rightarrow$" Let $\dim(V) < \infty$. Let β be a finite basis for V.

$W \overset{T \text{ is onto}}{=} R(T) = \text{span}(T(\beta)) \quad \therefore \dim(W) \leq n < \infty$.

"$\Leftarrow$" Let $\dim(W) < \infty$,
apply $T^{-1} \in \mathcal{L}(W, V)$ to show $\dim(V) < \infty$. $\square$

Let $\dim(V) = n < \infty$, $\beta = \{v_1, \cdots, v_n\}$ a basis for V. Then
W=span($\{T(v_1), \cdots, T(v_n)\}$)

Claim: $T(\beta) = \{T(v_1), \cdots, T(v_n)\}$ is l. indep.
($\because T$ is one-to-one).

If so, $T(\beta)$ is a basis for W. $\sharp T(\beta) = n \therefore$ dim(W)$= n =$dim(V).

Proof of claim: Let $\sum_{i=1}^{n} a_i T(v_i) = 0_v$ for $a_1, \cdots, a_n \in F$.

To show: $a_1 = \cdots = a_n = 0$.

$\because 0 = \sum_{i=1}^{n} a_i T(v_i) = T(\sum_{i=1}^{n} a_i v_i)$ $(\because T \in \mathcal{L})$

And T is one-to-one.

$\therefore \sum_{i=1}^{n} a_i v = 0$

$\because \beta = \{v_1, \cdots, v_n\}$ is l.indep.

$\therefore a_1 = \cdots = a_n = 0.$

Thm. Let $T \in \mathcal{L}(V, W)$, where V, W are finite-dimensional with ordered bases β, γ, respectively. Then

$$T \text{ is invertible } \textbf{iff } [T]_\beta^\gamma \text{ is invertible.}$$

Moreover,

$$[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}.$$

Proof. "$\Rightarrow$" Assume: T is invertible.
First, $\dim(V) = \dim(W)$ by lemma. Let $n = \dim(V) = \dim(W)$.
By $T^{-1}T = 1_V$,

$$I_{n\times n} = [I_V]_\beta^\beta = [T^{-1}T]_\beta^\beta = [T^{-1}]_\gamma^\beta [T]_\beta^\gamma.$$

Similarly, by $TT^{-1} = I_W$,

$$I_{n\times n} = [I_W]_\gamma^\gamma = [TT^{-1}]_\gamma^\gamma = [T]_\beta^\gamma [T^{-1}]_\gamma^\beta.$$

$\therefore [T]_\beta^\gamma$ is invertible, and $([T]_\beta^\gamma)^{-1} = [T^{-1}]_\gamma^\beta$. $\square$

"$\Leftarrow$" Assume: $A \overset{def}{=} [T]_\beta^\gamma$ is invertible (of finite size)

To show $T \in \mathcal{L}(V, W)$ is invertible. It suffices to show T is one to one.

Let $v_1, v_2 \in V$, and $T(v_1) = T(v_2)$.

$\Rightarrow [T(v_1)]_\gamma = [T(v_2)]_\gamma$

$\Rightarrow [T]_\beta^\gamma [v_1]_\beta = [T]_\beta^\gamma [v_2]_\beta$

$\Rightarrow [v_1]_\beta = [v_2]_\beta$ ($\because [T]_\beta^\gamma$ is invertible)

$\Rightarrow v_1 = v_2$. $\square$

Remark:

$$\begin{array}{ccc} V & \underset{T^{-1}}{\overset{T}{\rightleftarrows}} & W \\ \downarrow [\cdot]_\beta & & \downarrow [\cdot]_\gamma \\ \mathbb{F}^{\dim(V)} & \underset{[T^{-1}]_\gamma^\beta}{\overset{[T]_\beta^\gamma}{\rightleftarrows}} & \mathbb{F}^{\dim(W)} \end{array}$$

T is invertible $\Leftrightarrow$ $[T]_\beta^\gamma$ is invertible

$$V \overset{\exists U \in \mathcal{L}(W,V)}{\longleftarrow} W$$

$$\boxed{\begin{array}{c} \bullet \sum_{i=1}^n B_{i1} v_i \\ \bullet \sum_{i=1}^n B_{i2} v_i \\ \vdots \\ \bullet \sum_{i=1}^n B_{in} v_i \end{array}} \begin{array}{c} \longleftarrow \\ \longleftarrow \\ \vdots \\ \longleftarrow \end{array} \boxed{\begin{array}{c} \bullet w_1 \\ \bullet w_2 \\ \vdots \\ \bullet w_n \end{array}}$$

such that $U(w_j) = \sum_{i=1}^n B_{ij} v_i, j = 1, \cdots, n.$

By def.: $[B_{ij}]_{n\times n} = [U]_\gamma^\beta$. $\square$

Corollary. $T \in \mathcal{L}(V)$, where $\dim(V) < \infty$ and β is an ordered basis for V. Then,

$$T \text{ is invertible } \textbf{iff } [T]_\beta \text{ is invertible.}$$

Moreover, in this case,

$$[T^{-1}]_\beta = ([T]_\beta)^{-1}.$$

$$\begin{tikzcd}
V \arrow[r, shift left, "T"] \arrow[d, "{[\cdot]_\beta}"] & V \arrow[l, shift left, "T^{-1}"] \arrow[d, "{[\cdot]_\gamma}"] \\
\mathbb{F}^n \arrow[r, shift left, "{[T]_\beta}"] & \mathbb{F}^n \arrow[l, shift left, "{[T^{-1}]_\beta}"]
\end{tikzcd}$$

Def. Let V, W: v.s. Then, V is **isomorphic** to W if there is an invertible $T \in \mathcal{L}(V, W)$.

In this case, T is called an **isomorphism** from V onto W.

Thm. Let V, W be finite-dimensional v.s.. Then,

$$V \text{ is isomorphic to } W \textbf{ iff } \dim(V) = \dim(W).$$

Proof. "$\Rightarrow$" Assume: V is isomorphic to W.
$\therefore \exists$ an isomorphism $T \in \mathcal{L}(V, W)$
$\because T$ is invertible $\therefore$ By the previous lemma, $\dim(V) = \dim(W)$. $\square$

"$\Leftarrow$" Assume: $\dim(V) = \dim(W) \stackrel{def}{=} n < \infty$. Let

$$\beta = \{v_1, \cdots, v_n\} : \text{basis for V}$$
$$\gamma = \{w_1, \cdots, w_n\} : \text{basis for } W$$

Then, $\exists! T \in \mathcal{L}(V, W)$ such that $T(v_i) = w_i, i = 1, \cdots, n$.
Then $R(T) = \text{span}(T(\beta)) = \text{span}(\gamma) = W$
$\therefore$ T is onto, hence one-to-one (dim(V)=dim(W)<∞)
$\therefore$ T is bijective. So $T \in \mathcal{L}(V, W)$ is invertible. So T is an isomorphism.
$\therefore$ V is isomorphic to W. $\square\square$

Corollary. Let V be a v.s. over $\mathbb{F}$. Then

$$V \text{ is isomorphic to } \mathbb{F}^n \textbf{ iff } \dim(V) = n.$$

e.g. set dim(V)=n and β is an o.b. for V.

Write the standard representation of V under β as $[\cdot]_\beta = \phi_\beta$.

Then take any $v \in V$, see $[v]_\beta \in \mathbb{F}^n$ where $[v]_\beta$ is β-coordinate of $v \in V$.

The $[\cdot]_\beta : V \to \mathbb{F}^n$ is isomorphism.

Thm. Let V, W be finite-dimensional v.s. over $\mathbb{F}$ with $\dim(V) = n$, $\dim(W) = m$, and ordered bases β, γ, resp.

$$\begin{tikzcd}
V \arrow[r, shift left, "T \in \mathcal{L}(V,W)"] \arrow[d, "{[\cdot]_\beta}"] & W \arrow[l, shift left, "T^{-1}"] \arrow[d, "{[\cdot]_\gamma}"] \\
\mathbb{F}^n \arrow[r, shift left, "{[T]_\beta^\gamma \in M_{m\times n}(\mathbb{F})}"] & \mathbb{F}^m \arrow[l, shift left]
\end{tikzcd}$$

Then, the mapping

$$\Phi : \mathcal{L}(V, W) \to M_{m\times n}(\mathbb{F})$$
$$T \mapsto \Phi(T) = [T]_\beta^\gamma$$

is an isomorphism.
(i.e. $\mathcal{L}(V, W)$ is isomorphic to $M_{m\times n}(\mathbb{F})$).

This tells: (1) $\mathcal{L}(V, W)$ is finite-dimensional.
(2) $\dim(\mathcal{L}(V, W)) = \dim(M_{m\times n}(\mathbb{F})) = mn$.

Proof.: 1°. Φ is linear.

2° Φ is one-to-one.

$\Phi(T_1) = \Phi(T_2)$ i.e. $[T_1]_\beta^\gamma = [T_2]_\beta^\gamma$ to show: $T_1 = T_2$

take $v \in V$, $[T_1(v)]_\gamma = [T_1]_\beta^\gamma [v]_\beta$, $[T_2(v)]_\gamma = [T_2]_\beta^\gamma [v]_\beta$

$\because [T_1(v)]_\gamma = [T_2(v)]_\gamma \therefore T_1(v) = T_2(v)$.

3°. Φ is onto. let $A = (a_{ij})_{m\times n} \in M_{m\times n}(\mathbb{F})$.
To show: $\exists T \in \mathcal{L}(V, W)$ s.t. $A = \Phi(T) = [T]_\beta^\gamma$.

Indeed, $\beta = \{v_1, \cdots, v_n\}$ o.b. for V and $\gamma = \{w_1, \cdots, w_m\}$ o.b. for W.

Then, $\exists! T \in \mathcal{L}(V, W)$ such that $T(v_j) = \sum_{i=1}^m a_{ij} w_i, 1 \le j \le n$.

$\therefore A = [T]_\beta^\gamma = \Phi(T)$, i.e. T is onto.

Def. Let V be a v.s. over $\mathbb{F}$ with $\dim(V) = n$, and β be an ordered basis for V. The map

$$\begin{aligned} \Phi_\beta :\ & V \to \mathbb{F}^n \\ & v \mapsto \Phi_\beta(v) \stackrel{def.}{=} [v]_\beta \end{aligned}$$

is called the **standard representation** of V w.r.t. β.

Note: Φ_β is an isomorphism from V to $\mathbb{F}^n$. $\square$