Topic#6

Null space, range, and Dimension Theorem

Def. V, W: v.s. over $\mathbb{F}$. $T : V \to W$ linear.

$$N(T) \stackrel{def}{=} \{x \in V : T(x) = 0_W\}$$

is called the null space (or kernel) of T.

$$R(T) \stackrel{def}{=} \{T(x) : x \in V\} \subset W$$

is called the range (or image) of T.

Prop. $T : V \to W$ is linear. Then, $N(T)$ is a subspace of V, and $R(T)$ is a subspace of W.

Proof. $N(T)$ is a subspace of V. Indeed, $N(T) \subset V$, and
(1) $T(0_V) = 0_W$. $\therefore 0_V \in N(T)$
(2) Let $x, y \in N(T)$, $a \in \mathbb{F}$.
$T(x+y) = T(x) + T(y) = 0_W + 0_W = 0_W$
$T(ax) = aT(x) = a0_W = 0_W$
$\therefore x + y \in N(T), ax \in N(T)$. □

$R(T)$ is s subspace of W. Indeed, $R(T) \subset W$, and
(1) $T(0_V) = 0_W$. $\therefore 0_W \in R(T)$
(2) Let $x, y \in R(T)$, $a \in \mathbb{F}$. Then $\exists v, w \in V$, s.t.
$x = T(v), y = T(w)$.
$\therefore x + y = T(v) + T(w) = T(v+w)$ (T: linear)
with $v + w \in V$ ($v, w \in V, V : v.s$)
$ax = aT(v) = T(av)$ with $av \in V$
$\therefore x + y \in R(T), ax \in R(T)$. □□

e.g.: (1) $T_0 : V \to W$ (zero transf.):

$$N(T_0) = V, R(T_0) = \{0_N\}.$$

$I_V : V \to V$ (identity transf.):

$$N(I_V) = \{0_V\}, R(I_V) = V.$$

(2) $A \in M_{m\times n}(\mathbb{F})$, $L_A : \mathbb{F}^n \to \mathbb{F}^m$ (left-multiplication)
$N(L_A) = N(A)$: null space of A.
$R(L_A) = \mathcal{C}(A) : \mathcal{C}(A)$ is the column space of A. Note

$$Ax = (\square, \square, \cdots, \square)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\square + x_2\square + \cdots + x_n\square.$$

(3) $T : P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$, $f \in P_n(\mathbb{R}) \mapsto Tf \in P_{n-1}(\mathbb{R})$ by

$$Tf(x) = f'(x), \ \forall x \in \mathbb{R}.$$

$N(T) = \{$ const. poly. $\} = P_0(\mathbb{R})$
$R(T) = P_{n-1}(\mathbb{R})$.

Goal 1: Let $T \in \mathcal{L}(V, W)$, to find a spanning set of $R(T)$ in terms of a basis for V.

Thm: let $T \in \mathcal{L}(V, W)$ where V, W are v.s. and V is finite-dimensional. Let V has a basis $\beta = \{v_1, v_2, \cdots, v_n\}$. Then:

$$R(T) = \text{span}(T(\beta)) = \text{span}(\{T(v_1), T(v_2), \cdots, T(v_n)\}).$$

Proof. "$\supset$": $\beta \subset V$, $R(T) \supset T(\beta)$, $R(T)$ is a subpsace of W containing $T(\beta)$, and span$(T(\beta))$ is the smallest subspace of W containing $T(\beta)$. $\therefore R(T) \supset$span$(T(\beta))$. $\square$

"$\subset$": Let $w \in R(T)$. $\exists v \in V$, s.t. $w = T(v)$. β is a basis for V
$\therefore \exists! a_1, \cdots, a_n \in \mathbb{F}$, s.t. $v = \sum_{i=1}^{n} a_i v_i$. Then
$w = T(v) = T(\sum_{i=1}^{n} a_i v_i) = \sum_{i=1}^{n} a_i T(v_i) \in \text{span}(T(\beta))$.
Note w is linear combination of vectors in $T(\beta)$.
$\therefore R(T) \subset$span$(T(\beta))$. $\square\square$

Remark: Thm is also true even if β is infinite (countable or uncountable). □

Remark: $R(T) = \text{span}(\{T(v_1), \cdots, T(v_n)\})$.
When $T(\beta) = \{T(v_1), \cdots, T(v_n)\}$ is l. indep.?

Let $\sum_{i=1}^{n} a_i T(v_i) = 0$. Then, $T(\sum_{i=1}^{n} a_i v_i) = 0$.
Assume $N(T) = \{0\}$.
Then $\sum_{i=1}^{n} a_i v_i = 0$. $\therefore a_1 = \cdots = a_n = 0$.
This shows:
If $N(T) = \{0\}$,
then $T(\beta)$ is l. indep. and thus $T(\beta)$ is a basis for $R(T)$. □

e.g.: $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ is defined by

$$f \in P_2(\mathbb{R}) \mapsto Tf = \begin{pmatrix} f(1) - f(2) & 0 \\ 0 & f(0) \end{pmatrix}$$

1^o. $T \in \mathcal{L}(P_2(\mathbb{R}), M_{2\times 2}(\mathbb{R}))$ (i.e. T is linear)
2^o. $\beta = \{1, x, x^2\}$ a basis for $P_2(\mathbb{R})$

$$\begin{aligned} R(T) &= \text{span}(T(\beta)) \quad \text{(thm)} \\ &= \text{span}(\{T(1), T(x), T(x^2)\}) \\ &= \text{span}(\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1-2 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1^2 - 2^2 & 0 \\ 0 & 0^2 \end{pmatrix}\}) \\ &= \text{span}(\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\}) \end{aligned}$$

$\therefore \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ is a basis for $R(T)$, $\dim(R(T)) = 2$. $\square$

Goal 2: measure the size of subspaces $N(T), R(T)$ by their dimensions.

note:

• The larger $N(T)$ (its dim), the smaller $R(T)$ (its dim), for instance, $T = T_0$.

• The smaller $N(T)$ (its dim), the larger $R(T)$ (its dim), for instance, $T = I_V$.

Def. Let $T \in \mathcal{L}(V, W)$.

Assume $N(T), R(T)$ are finite-dimensional.

$$\text{nullity}(T) \stackrel{def}{=} \dim(N(T))$$

$$\text{rank}(T) \stackrel{def}{=} \dim(R(T))$$

Dimension Thm: Let $T \in \mathcal{L}(V, W)$, and V be finite-dimensional. Then,

$$\text{nullity}(T) + \text{rank}(T) = \dim(V).$$

Proof. Note $N(T)$ is a subspace of finite-dimensional V, $N(T)$ is finite-dimensional.

Assume: $n = \dim(V)$, $k = \dim(N(T))$, with $k \leq n$,

$\{v_1, \cdots, v_k\}$ is a basis for $N(T)$,

extend $\{v_1, \cdots, v_k\}$ to be a basis $\beta = \{v_1, \cdots, v_n\}$ for V.

To show: $\gamma \overset{def}{=} \{T(v_{k+1}), \cdots, T(v_n)\}$ is a basis for $R(T)$.

Indeed, 1°. $R(T) = \text{span}\gamma$. In fact, from the previous thm,

$R(T) = \text{span}(\{T(v_1), \cdots, T(v_n)\}) = \text{span}(\{T(v_{k+1}), \cdots, T(v_n)\})$

$= \text{span}\gamma$

$(\because T(v_i) = 0, 1 \leq i \leq k)$.

2°. γ is l. indep. In fact, let $\sum_{i=k+1}^{n} a_i T(v_i) = 0, a_i \in \mathbb{F}$,
then $T(\sum_{i=k+1}^{n} a_i v_i) = 0$ ($\because$ T is linear)
$\therefore \sum_{i=k+1}^{n} a_i v_i \in N(T) = \text{span}(\{v_1, \cdots, v_k\})$
$\therefore \exists b_1, b_2, \cdots, b_k \in \mathbb{F}$, s.t. $\sum_{i=k+1}^{n} a_i v_i = \sum_{i=1}^{k} b_i v_i$
i.e. $\sum_{i=1}^{k}(-b_i)v_i + \sum_{i=k+1}^{n} a_i v_i = 0$
$\therefore a_{k+1} = \cdots = a_n = 0$ ($\because$ $\beta = \{v_1, \cdots, v_n\}$ is a basis for V)
$\therefore \gamma$ is l. indep. □

Following the previous example:

$$\underbrace{\text{nullity}(T)}_{=\dim(N(T))} + \underbrace{\text{rank}(T)}_{\dim(R(T))=2} = \underbrace{\dim(P_2(\mathbb{R}))}_{=3}$$

$\therefore \dim(N(T)) = 1.$

It is also direct to compute:

$$\begin{aligned}
&Tf = 0\\
&\Leftrightarrow f(0) = 0, f(1) = f(2), f = a_0 + a_1x + a_2x^2\\
&\Leftrightarrow a_0 = 0, a_1 + a_2 = 2a_1 + 4a_2\\
&\Leftrightarrow a_0 = 0, a_1 + 3a_2 = 0\\
&\Leftrightarrow f(x) = -3a_2x + a_2x^2 = a_2(-3x + x^2)
\end{aligned}$$

$\therefore \dim(N(T)) = 1$

Goal 3: Let $T \in \mathcal{L}(V, W)$, find relations between

$$\boxed{T \text{ is one-to-one or onto}} \longleftrightarrow \boxed{N(T),\ R(T) \text{ \& their dimensions}}$$

Thm#1: $T \in \mathcal{L}(V, W)$. Then T is one-to-one **iff** $N(T) = \{0\}$.

Proof. "$\Rightarrow$" Let T be one-to-one, it is sufficent to show: $N(T) \subset \{0\}$.
Let $x \in N(T)$.
$\therefore\ T(x) = 0 = T(0_V),\ \therefore x = 0_V$ ($\because$ T is one-to-one) $\square$

"$\Leftarrow$" Let $N(T) = \{0\}$, to show: T is one-to-one.
Let $T(x) = T(y), x, y \in V$.
$\therefore 0 = T(x) - T(y) = T(x - y)$ (T: linear)
$\therefore x - y = 0, (\because N(T) = \{0\})$
i.e. $x = y$, then T is one-to-one. $\square\square$

Thm#2: Let $T \in \mathcal{L}(V, W)$ with $\dim(V) = \dim(W) < \infty$.
Then the following are equivelent:
(a) T is one-to-one.
(b) T is onto.
(c) $\text{rank}(T) = \dim(V)$.
(d) $\text{nullity}(T) = 0$.

Proof. to show $(a) \Leftrightarrow (d) \Leftrightarrow (c) \Leftrightarrow (b)$:
$(a) \Leftrightarrow (d)$: T is ont-to-one $\Leftrightarrow N(T) = 0 \Leftrightarrow \dim(N(T)) = 0$
$(d) \Leftrightarrow (c)$: due to dimension thm: $\text{nullity}(T) + \text{rank}(T) = \dim(V)$
$(c) \Leftrightarrow (b)$: $\text{rank}(T) = \dim(V)$
$\Leftrightarrow \dim(R(T)) = \dim(W)$
$\Leftrightarrow R(T) = W$ ("$\Leftarrow$" obvious, "$\Rightarrow$" $R(T)$ is a subspace of W. $R(T)$ has the same dim as W.)
$\Leftrightarrow T$ is onto $\square$

e.g.: Construct $T \in \mathcal{L}(V, W)$ with $\dim(V) \neq \dim(W)$ s.t. T is one-to-one but not onto.

$T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ is defined by

$$f(x) \in P_2(\mathbb{R}) \mapsto T(f(x)) \in P_3(\mathbb{R}) : T(f(x)) \overset{def}{=} 2f'(x) + \int_0^x 3f(t)dt.$$

1°. $T \in \mathcal{L}(P_2(\mathbb{R}), P_3(\mathbb{R}))$ (verify this as an exercise).

2°. $\beta = \{1, x, x^2\}$: basis for $P_2(\mathbb{R})$
$T(\beta) = \{T(1), T(x), T(x^2)\} = \{3x, 2 + \frac{3}{2}x^2, 4x + x^3\}$
(It is l. indep. Why?) A basis for $R(T)$
$\therefore R(T)$ =span($T(\beta)$)=span($\{3x, 2 + \frac{3}{2}x^2, 4x + x^3\}$)
$\therefore$ rank(T)=dim($R(T)$)=3<dim(P_3)=4
$\therefore T$ is not onto

3°. Dimension thm:

nullity$(T) = \dim(N(T)) = \dim(P_2(\mathbb{R})) - \text{rank}(T) = 3 - 3 = 0$
$\therefore T$ is one-to-one.