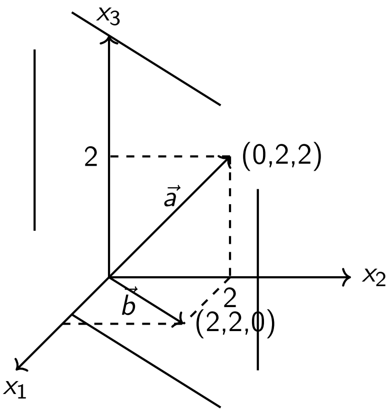
Topic#2
Subspace



$$\mathcal{P} \overset{def.}{=} \text{the plane through origin containing } \vec{a} \ \& \ \vec{b}$$

$$= \{s\vec{a} + t\vec{b} : s, t \in \mathbb{R}\}$$

$\mathcal{P}$ is a "subspace" of $\mathbb{R}^3$

Let $(V, +, \cdot)$ be a v.s. over $\mathbb{F}$.

Def. U is a **subspace** of V if

(1) $U \subset V$

(2) U is a v.s. over $\mathbb{F}$ with the same "$+$" and "$\cdot$".

e.g.: use def. to check the previous example.

Here is an alternative easy way to check if U is a subspace.

Prop. $(V, +, \cdot)$ v.s. over $\mathbb{F}$, $U \subset V$.

Then U is a subspace **iff**

(1) $0 \in U$. (zero of V is in U)
(2) $u, v \in U \Rightarrow u + v \in U$. ($U$ is closed under "$+$")
(3) $a \in F$, $u \in U \Rightarrow au \in U$.($U$ is closed under "$\cdot$")

Proof.

"$\Rightarrow$" Assume U is a subspace and hence is a vector space. We see (2) and (3) are satisfied by def. of v.s.

For (1), it means to show:

$$0 = o_V \in U,$$

namely, zero of V is also in U (hence zero of V is also zero of U by uniqueness of zero in U).

Since $o_U \in U$, it suffices to show $o_V = o_U$.

In fact, for $0 \in F$ and $o_U \in U$, $0o_U \in U$ by (3). Note

$$0o_U = o_V$$

since $0x = o_V$ for any $x \in V$ & $o_U \in U \subset V$. Hence $o_V \in U$. $\square$

"$\Leftarrow$" to show U is a v.s., i.e.
(a). $+, \cdot$: well-defined (by (2)&(3))
(b). VS1)-(VS8) satisfied for $(U, +, \cdot)$

✓ (VS1) $u + v = v + u$, $\forall u, v \in U$
✓ (VS2) $(u + v) + w = u + (v + w)$, $\forall u, v, w \in U$
? (VS3) $\exists 0 \in U$ s.t. $u + 0 = u$, $\forall u \in U$
? (VS4) $\forall u \in U$ $\exists w \in U$ s.t. $u + w = 0$
✓ (VS5) $1u = u$, $\forall u \in U$
✓ (VS6) $(ab)u = a(bu)$, $\forall a, b \in \mathbb{F}$, $\forall u \in U$
✓ (VS7) $a(u + v) = av + au$, $\forall a \in \mathbb{F}$, $\forall u, v \in U$
✓ (VS8) $(a + b)u = au + bu$, $\forall a, b \in \mathbb{F}$, $\forall u \in U$

(1)$\Rightarrow$(VS3) true.

Proof of (VS4):
Let $u \in U \subset V$, $\exists w = -u \in V$ s.t. $u + w = 0$.
note: $w = -u = (-1)u \in U$ by (3). □□

Remark: $(V, +, \cdot)$ is v.s. over $\mathbb{F}$.

$\{0\}$ is the smallest subspace of V
V is the largest subspace of V

Prooof. $0 \in \{0\}$. $0 + 0 = 0$, $a0 = 0 \in \{0\}$, $\forall a \in \mathbb{F}$.

Warning: $\emptyset \subset V$, but $\emptyset$ is NOT a subspace
(why?) ($\because 0 \in U$ $\therefore$ any subspace should be non-empty)

Examples:

(1) $\{(x_1, x_2, x_3) \in \mathbb{F}^3, x_1 + 2x_2 - 3x_3 = a\}$ is a subspace of $\mathbb{F}^3$ iff $a = 0$

Check: $\Leftarrow$ Let $a = 0$, then...(check 3 conditions in the Prop.)

$\Rightarrow$ It contains the zero vector, then $a = 0$. $\square$

(2) $\mathbb{R}^{[0,1]}$ = set of all real-valued functions on $[0,1]$

check: $\mathbb{R}^{[0,1]}$ is v.s. over $\mathbb{R}$.

$$C([0,1]) \stackrel{def}{=} \{f \in \mathbb{R}^{[0,1]} \mid f \text{ is continuous}\}$$

is a subspace of $\mathbb{R}^{[0,1]}$

Check:

(a): $C([0,1]) \subset \mathbb{R}^{[0,1]}$

(b):
(1) $0 \in C([0,1])$,
(2) $f, g \in C([0,1]) \Rightarrow f + g \in C([0,1])$
(3) $a \in \mathbb{R}, f \in C([0,1]) \Rightarrow af \in C([0,1])$ □

Note: $\{f \in C([0,1]) \to \mathbb{R} \mid f \text{ continuous in } [0,1]$
f differentiable in $(0,1)$ $\}$

is a subspace of $C([0,1])$

(3) $\mathbb{C}^\infty \overset{def}{=} \{(z_1, z_2, \cdots) : z_i \in \mathbb{C}\}$ v.s.

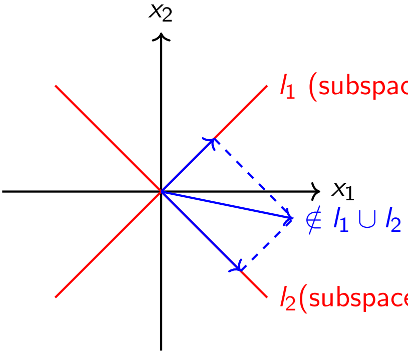
$$\{(z_1, z_2, \cdots) \in \mathbb{C}^\infty, \mid \lim_{i\to\infty} z_i = 0\}$$

is a subspace of $\mathbb{C}^\infty$

Question: $(V, +, \cdot)$ v.s. over $\mathbb{F}$, U, W subspaces

$U \cap W$, $U \cup W$ subspace?

$V = \mathbb{R}^2$ (v.s.)



$l_1 \cap l_2 = \{0\}$ is a vector space. But, $l_1 \cup l_2$ is not a vector space

> **Prop.** $(V, +, \cdot)$ v.s. over $\mathbb{F}$, U, W are subspaces of V, then
>
> $U \cap W$ is a subspace of V.

Proof.

(1) $\because U, W$ are subspaces of V, $\therefore U \subset V, W \subset V$, $\therefore U \cap W \subset V$
(2) Let 0 be the zero vector of V, then $0 \in U$ and $0 \in W$,
$\therefore 0 \in U \cap W$
(3) Let $u, v \in U \cap W$, $\therefore u \in U \cap W, v \in U \cap W$
$\therefore u, v \in U, u, v \in W$
$\because U$ is a subspace, $\therefore u + v \in U$. Similarly, $u + v \in W$.
$\therefore u + v \in U \cap W$
(4) Let $a \in \mathbb{F}$, $v \in U \cap W$ to show: $av \in U \cap W$.
Indeed, $\because v \in U \cap W$, $\therefore v \in U, v \in W$
$\because U, W$ are subspaces of V, $\therefore av \in U$ and $av \in W$
$\therefore av \in U \cap W$. □

Remark Similarly, one can show:

Claim: Let $\{U_i\}_{i \in I}$ be a collection of subspaces of a v.s. V where I is a set of index (even infinitely uncountable), then $\cap_{i \in I} U_i$ is a subspace of V.