MATH2040A Homework 2
Reference Solutions

1 Compulsory Part

1.4.10. Show that if

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \text{and} \quad M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then the span of $\{ M_1, M_2, M_3 \}$ is the set of all symmetric 2×2 matrices.

Solution: Denote the set of 2×2 symmetric matrices by Sym_2. Trivially we have $\mathrm{Sym}_2 \subseteq M_{2\times 2}$.

Let $A \in \mathrm{Sym}_2$. We may assume that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{F}$. Since A is symmetric, we have that $b = c$. Then $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = aM_1 + dM_2 + bM_3 \in \mathrm{Span}(\{ M_1, M_2, M_3 \})$. As A is arbitrary, we have that $\mathrm{Sym}_2 \subseteq \mathrm{Span}(\{ M_1, M_2, M_3 \})$.

Trivially, $M_1, M_2, M_3 \in \mathrm{Sym}_2$. As Sym_2 is a subspace of $M_{2\times 2}$ (see the remark after Theorem 1.3 in textbook), we have by Theorem 1.5.6 that $\mathrm{Span}(\{ M_1, M_2, M_3 \}) \subseteq \mathrm{Sym}_2$.

These two imply that $\mathrm{Sym}_2 = \mathrm{Span}(\{ M_1, M_2, M_3 \})$.

1.4.11. Prove that $\mathrm{Span}(\{ x \}) = \{ ax : a \in \mathbb{F} \}$ for any vector x in a vector space. Interpret this result geometrically in $\mathbb{R}^3$.

Solution: Let $y \in \mathrm{Span}(\{ x \})$. Then $y = \sum_{i=1}^n a_i x_i$ for some $n \in \mathbb{N}$, $a_i \in \mathbb{F}$, and $x_i \in \{x\}$. As $\{x\}$ contains only one element, $x_i = x$ for all i. So $y = \sum_{i=1}^n a_i x_i = \sum_{i=1}^n a_i x = (\sum_{i=1}^n a_i) \cdot x \in \{ ax : a \in \mathbb{F} \}$. As y is arbitrary, $\mathrm{Span}(\{ x \}) \subseteq \{ ax : a \in \mathbb{F} \}$.

Let $y \in \{ ax : a \in \mathbb{F} \}$. Then for some $a \in \mathbb{F}$, $y = ax$. As $x \in \{x\}$, y is a linear combination of the elements in $\{x\}$, and so $y \in \mathrm{Span}(\{ x \})$. As y is arbitrary, $\{ ax : a \in \mathbb{F} \} \subseteq \mathrm{Span}(\{ x \})$.

Therefore, $\mathrm{Span}(\{ x \}) = \{ ax : a \in \mathbb{F} \}$.

If $x = 0$, the span of the singleton $\{x\} = \{0\}$ is just the origin $\{0\}$.

If $x \neq 0$, the span of $\{x\}$ is the (unique) straight line that passes through both the origin and x.

Note

You cannot simply state that $\mathrm{Span}(\{ x \})$ and $\{ ax : a \in \mathbb{F} \}$ are equal "by definition", as by definition the span of a set S is "the set consisting of all linear combinations of the vectors in S" and does not give you an explicit expression on the set. You are supposed to show the expression. This is almost trivial, but you *still* need to show it.

1.4.13. Show that if S_1 and S_2 are subsets of a vector space V such that $S_1 \subseteq S_2$, then $\mathrm{Span}(S_1) \subseteq \mathrm{Span}(S_2)$. In particular, if $S_1 \subseteq S_2$ and $\mathrm{Span}(S_1) = V$, deduce that $\mathrm{Span}(S_2) = V$.

Solution: Let $v \in \mathrm{Span}(S_1)$. Then there exists $n \in \mathbb{Z}^+$, $a_1, \ldots, a_n \in \mathbb{F}$ and $v_1, \ldots, v_n \in V$ such that $v = \sum_{i=1}^n a_i v_i$. Since $S_1 \subseteq S_2$, $v_i \in S_2$ for all i. This implies that v is also a linear combination of vectors in S_2 and so $v \in \mathrm{Span}(S_2)$. As v is arbitrary, $\mathrm{Span}(S_1) \subseteq \mathrm{Span}(S_2)$.

Suppose now that $\mathrm{Span}(S_1) = V$. The $V = \mathrm{Span}(S_1) \subseteq \mathrm{Span}(S_2)$. As $S_2 \subseteq V$, by Theorem 1.5 we also have $\mathrm{Span}(S_2) \subseteq V$. So $\mathrm{Span}(S_2) = V$.

Note

Do not assume any of the sets is finite. In particular, do not write $S_1 = \{ v_1, \ldots, v_n \}$ for some $n \in \mathbb{N}$ unless you know it is a finite set.

1.4.14. Show that if S_1 and S_2 are arbitrary subsets of a vector space V, then $\mathrm{Span}(S_1 \cup S_2) = \mathrm{Span}(S_1) + \mathrm{Span}(S_2)$.

Solution:

(a) Let $v \in \mathrm{Span}(S_1 \cup S_2)$. Then there exists $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{F}$ and $v_1, \ldots, v_n \in S_1 \cup S_2$ such that $v = \sum_{i=1}^{n} a_i v_i$, with an empty sum representing 0.

If $n = 0$, $v = 0 \in \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$, and so $v = 0 + 0 \in \mathrm{Span}(S_1) + \mathrm{Span}(S_2)$. In the following we will assume that $n \geq 1$.

As each of $v_i \in S_1 \cup S_2$, either $v_i \in S_1$ or $v_i \in S_2 \setminus S_1$ but not both. By symmetry of the indices, we may assume that $v_1, \ldots, v_k \in S_1$ and $v_{k+1}, \ldots, v_n \in S_2 \setminus S_1$ for some $k \in \{ 0, \ldots, n \}$, with $k = 0$ representing the case that $v_i \in S_2 \setminus S_1$ for all i and $k = n$ representing that $v_i \in S_1$ for all i. This implies that $\sum_{i=1}^{k} a_i v_i \in \mathrm{Span}(S_1)$ and $\sum_{j=k+1}^{n} a_j v_j \in \mathrm{Span}(S_2)$ as they are the linear combinations of the elements of the corresponding sets, with an empty sum representing 0. So $v = \left(\sum_{i=1}^{k} a_i v_i \right) + \left(\sum_{j=k+1}^{n} a_j v_j \right) \in \mathrm{Span}(S_1) + \mathrm{Span}(S_2)$.

As v is arbitrary, $\mathrm{Span}(S_1 \cup S_2) \subseteq \mathrm{Span}(S_1) + \mathrm{Span}(S_2)$.

(b) Let $v \in \mathrm{Span}(S_1) + \mathrm{Span}(S_2)$. Then there exists $x \in \mathrm{Span}(S_1), y \in \mathrm{Span}(S_2)$ such that $v = x + y$. Then there exist $n_1, n_2 \in \mathbb{N}$, $a_1, \ldots, a_{n_1}, b_1, \ldots, b_{n_2} \in \mathbb{F}$ and $u_1, \ldots, u_{n_1} \in S_1$, $w_1, \ldots, w_{n_2} \in S_2$ such that $x = \sum_{i=1}^{n_1} a_i u_i$, $y = \sum_{j=1}^{n_2} b_j w_j$, with the empty sum representing 0. So $v = x + y = \sum_{i=1}^{n_1} a_i u_i + \sum_{j=1}^{n_2} b_j w_j$ is a linear combination of vectors in $S_1 \cup S_2$, and so $v \in \mathrm{Span}(S_1 \cup S_2)$.

As v is arbitrary, $\mathrm{Span}(S_1) + \mathrm{Span}(S_2) \subseteq \mathrm{Span}(S_1 \cup S_2)$.

Therefore, $\mathrm{Span}(S_1 \cup S_2) = \mathrm{Span}(S_1) + \mathrm{Span}(S_2)$.

Note

You may also discuss the cases of the empty sets and empty sums separately.

1.4.15. Let S_1 and S_2 be subsets of a vector space V. Prove that $\mathrm{Span}(S_1 \cap S_2) \subseteq \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$. Give an example in which $\mathrm{Span}(S_1 \cap S_2)$ and $\mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$ are equal and one in which they are unequal.

Solution: Suppose $S_1 \cap S_2 = \emptyset$. Then $\mathrm{Span}(S_1 \cap S_2) = \{0\}$ and $\{0\} \subseteq \mathrm{Span}(S_1)$, $\{0\} \subseteq \mathrm{Span}(S_2)$, so $\mathrm{Span}(S_1 \cap S_2) = \{0\} \subseteq \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$.

Suppose now that $S_1 \cap S_2 \neq \emptyset$. In particular, neither S_1, S_2 is empty. Let $v \in \mathrm{Span}(S_1 \cap S_2)$. Then there exists $n \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{F}$, $v_1, \ldots, v_n \in S_1 \cap S_2$ such that $v = \sum_{i=1}^{n} a_i v_i$. Since $v_i \in S_1 \cap S_2$ for all i, we have $v_i \in S_1$ and $v_i \in S_2$ for all i. This implies that $v = \sum_{i=1}^{n} a_i v_i$ is linear combination of vectors from both S_1 and S_2, and so $v \in \mathrm{Span}(S_1)$, $v \in \mathrm{Span}(S_2)$. Hence $v \in \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$.

As v is arbitrary, $\mathrm{Span}(S_1 \cap S_2) \subseteq \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$.

We consider $V = \mathbb{R}$ with the usual $\mathbb{R}$-vector space structure to look for examples:

1. (Equal) Let $S_1 = S_2 = \emptyset$. Then $\mathrm{Span}(S_1 \cap S_2) = \mathrm{Span}(\emptyset) = \{0\} = \{0\} \cap \{0\} = \mathrm{Span}(S_1) \cap \mathrm{Span}(S_2)$.
2. (Unequal) Let $S_1 = \{ 1 \}$, $S_2 = \{ 2 \}$. Then $\mathrm{Span}(S_1 \cap S_2) = \mathrm{Span}(\emptyset) = \{0\} \neq \mathbb{R} = \mathbb{R} \cap \mathbb{R} = \mathrm{Span}(\{ 1 \}) \cap \mathrm{Span}(\{ 2 \})$.

1.5.9. Let u and v be distinct vectors in a vector space V. Show that $\{ u, v \}$ is linearly dependent if and only if u or v is a multiple of the other.

Solution:

(a) Suppose $\{ u, v \}$ is linear dependent. Then there exists $a, b \in \mathbb{F}$ not all zero such that $au + bv = 0$.

i. If $a \neq 0$, we have that $u = -\frac{b}{a}v$ is a multiple of v

ii. If $b \neq 0$, we have that $v = -\frac{a}{b}u$ is a multiple of u

Hence u or v is a multiple of the other.

(b) Suppose on the other hand that u or v is the multiple of the other.

i. If $u = cv$ for some $c \in \mathbb{F}$, then $1 \cdot u - cv = 0$ with $1, c \in \mathbb{F}$ being scalars not all zero

ii. If $v = cu$ for some $c \in \mathbb{F}$, then $c \cdot u - 1v = 0$ with $1, c \in \mathbb{F}$ being scalars not all zero

In both cases, there exists scalars $a, b \in \mathbb{F}$ not all zero such that $au + bv = 0$. This implies that $\{ u, v \}$ is linear dependent.

Hence $\{ u, v \}$ is linear dependent if and only if u, v is the multiple of the other.

1.5.13. Let V be a vector space over a field of characteristic not equal to 2.

(a) Let u and v be distinct vectors in V. Prove that $\{ u, v \}$ is linearly independent if and only if $\{ u+v, u-v \}$ is linearly independent.

(b) Let u, v, w be distinct vectors in V. Prove that $\{ u, v, w \}$ is linearly independent if and only if $\{ u+v, u+w, v+w \}$ is linearly independent.

Solution:

(a) i. Suppose $\{ u, v \}$ is linearly independent. Let $a, b \in \mathbb{F}$ be such that $a(u+v) + b(u-v) = 0$.
Then $(a+b)u + (a-b)v = 0$. By assumption, $a+b = a-b = 0$, so $2a = 2b = 0$. As the characteristic of $\mathbb{F}$ is not 2, $a = b = 0$.
This implies that $\{ u+v, u-v \}$ is linearly independent.

ii. Suppose $\{ u+v, u-v \}$ is linearly independent. Let $a, b \in \mathbb{F}$ be such that $au + bv = 0$.
Since the characteristic of $\mathbb{F}$ is not 2, $\frac{1}{2}$ exists in $\mathbb{F}$ and so $\frac{a+b}{2}, \frac{a-b}{2} \in \mathbb{F}$. Then $\frac{a+b}{2}(u+v) + \frac{a-b}{2}(u-v) = \left(\frac{a+b}{2} + \frac{a-b}{2}\right)u + \left(\frac{a+b}{2} - \frac{a-b}{2}\right)v = au + bv = 0$.
This implies that $\frac{a+b}{2} = \frac{a-b}{2} = 0$ and so $a = \frac{a+b}{2} + \frac{a-b}{2} = 0$, $b = \frac{a+b}{2} - \frac{a-b}{2} = 0$. Hence, $\{ u, v \}$ is linearly independent.

Hence $\{ u, v \}$ is linearly independent if and only if $\{ u+v, u-v \}$ is.

(b) i. Suppose $\{ u, v, w \}$ is linearly independent. Let $a, b, c \in \mathbb{F}$ be such that $a(u+v) + b(u+w) + c(v+w) = 0$.
Then $(a+b)u + (a+c)v + (b+c)w = 0$. By assumption, $a+b = a+c = b+c = 0$, so $2(a+b+c) = 0$. As the characteristic of $\mathbb{F}$ is not 2, $a+b+c = 0$, which gives $a = b = c = 0$.
This implies that $\{ u+v, u+w, v+w \}$ is linearly independent.

ii. Suppose $\{ u+v, u+w, v+w \}$ is linearly independent. Let $a, b, c \in \mathbb{F}$ be such that $au + bv + cw = 0$.
Since the characteristic of $\mathbb{F}$ is not 2, $\frac{1}{2}$ exists in $\mathbb{F}$ and so $\frac{a+b-c}{2}, \frac{a-b+c}{2}, \frac{-a+b+c}{2} \in \mathbb{F}$. Then $\frac{a+b-c}{2}(u+v) + \frac{a-b+c}{2}(u+w) + \frac{-a+b+c}{2}(v+w) = au + bv + cw = 0$.
This implies that $\frac{a+b-c}{2} = \frac{a-b+c}{2} = \frac{-a+b+c}{2} = 0$ and so $a = \frac{a+b-c}{2} + \frac{a-b+c}{2} = 0$, $b = \frac{a+b-c}{2} + \frac{-a+b+c}{2} = 0$, $c = \frac{a-b+c}{2} + \frac{-a+b+c}{2} = 0$. So $\{ u, v, w \}$ is linearly independent.

Hence $\{ u, v, w \}$ is linearly independent if and only if $\{ u+v, u+w, v+w \}$ is.

Note

To show that $\{ u, v \}$ is linearly independent assuming $\{ u+v, u-v \}$ is, you cannot consider only the coefficients of form $a+b, a-b$ and claim that $0 = (a+b)u + (a-b)v = a(u+v) + b(u-v)$ implies that $\{ u, v \}$ is linearly independent because $a = b = 0$ and so $a+b = a-b = 0$. You would have to show that this particular selection of coefficients $(a+b, a-b)$ is sufficient to cover all possible choices of coefficients (or equivalently $\{ (a+b, a-b) : a, b \in \mathbb{F} \} = \mathbb{F} \times \mathbb{F}$), which essentially is the same proof as presented here. Similar for part (b).

Some of you have asked about how the coefficients $\frac{a+b}{2}, \frac{a-b}{2}$ and $\frac{a+b-c}{2}, \frac{a-b+c}{2}, \frac{-a+b+c}{2}$ are found. If you go through the proof, you can notice that these coefficients are the (only) solutions that recover the original linear combinations. To be precise,

- $c = \frac{a+b}{2}, d = \frac{a-b}{2}$ is the only pair of coefficients such that $au + bv = c(u+v) + d(u-v)$ (or equivalently $(-a+c+d)u + (-b+c-d)v = 0$) holds for all vectors u, v and scalars a, b

- $d = \frac{a+b-c}{2}, e = \frac{a-b+c}{2}, f = \frac{-a+b+c}{2}$ is the only pair of coefficients such that $au + bv + cw = d(u+v) + e(u+w) + f(v+w)$ (or equivalently $(-a+d+e)u + (-b+d+f)v + (-c+e+f)w = 0$) holds for all vectors u, v, w and scalars a, b, c

as you can verify by solving the corresponding linear system. In general, you have the following result:

Lemma. *Let V be a vector space, $v_1, \ldots, v_n \in V$, $w_i = \sum_{j=1}^{n} c_{ij} v_j$ for $i \in \{ 1, \ldots, n \}$ where $c_{ij} \in \mathbb{F}$ for $i, j \in \{ 1, \ldots, n \}$ be scalars. Then for scalars $a_1, \ldots, a_n \in \mathbb{F}$ and $b_i = \sum_{j=1}^{n} c_{ji} a_j$ for $i \in \{ 1, \ldots, n \}$, $\sum_{i=1}^{n} a_i w_i = 0$ if and only if $\sum_{i=1}^{n} b_i v_i = 0$*

We left the detailed proof for you as an (easy) exercise, but by abusing notation you can see how obvious this lemma is:

$$\begin{pmatrix} a_1 & \ldots & a_n \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} a_1 & \ldots & a_n \end{pmatrix} \begin{pmatrix} c_{11} & \ldots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \ldots & c_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$= \left(\begin{pmatrix} c_{11} & \ldots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \ldots & c_{nn} \end{pmatrix}^{\mathsf{T}} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right)^{\mathsf{T}} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} b_1 & \ldots & b_n \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

1.5.15. Let $S = \{ u_1, u_2, \ldots, u_n \}$ be a finite set of vectors. Prove that S is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \text{Span}(\{ u_1, \ldots, u_k \})$ for some $k \in \{ 1, \ldots, n-1 \}$.

Solution:

(a) Suppose S is linearly dependent. Then there exist $a_1, \ldots, a_n \in \mathbb{F}$ not all zero be such that $\sum_{i=1}^{n} a_i u_i = 0$.

Let $k \in \{ 0, \ldots, n-1 \}$ be minimal integer such that $a_{k+1} \neq 0$ and $a_i = 0$ for all $i \in \{ 1, \ldots, n \}$ with $i > k+1$. Since $a_1, \ldots, a_n$ are not all zero, such k must exist in $\{ 0, \ldots, n-1 \}$.

i. If $k = 0$, $a_1 \neq 0$ and $a_2 = \ldots = a_n = 0$, so $0 = \sum_{i=1}^{n} a_i u_n = a_1 u_1$. As $a_1 \neq 0$, $u_1 = 0$

ii. If $k \geq 1$, $0 = \sum_{i=1}^{n} a_i u_i = \sum_{i=1}^{k} a_i u_i + a_{k+1} u_{k+1}$, so $u_{k+1} = \sum_{i=1}^{k} -\frac{a_i}{a_{k+1}} u_i \in \text{Span}(\{ u_1, \ldots, u_k \})$.

Hence, $u_1 = 0$ or $u_{k+1} \in \text{Span}(\{ u_1, \ldots, u_k \})$ for some $k \in \{ 1, \ldots, n-1 \}$.

(b) Suppose $u_1 = 0$ or $u_{k+1} \in \text{Span}(\{ u_1, \ldots, u_k \})$ for some $k \in \{ 1, \ldots, n-1 \}$.

i. If $u_1 = 0$, then $1u_1 + \sum_{i=2}^{n} 0u_i = 0$ with the scalars not all zero, so S is linearly dependent.

ii. If $u_{k+1} \in \text{Span}(\{ u_1, \ldots, u_k \})$ for some $k \in \{ 1, \ldots, n-1 \}$. Then for some $a_1, \ldots, a_k \in \mathbb{F}$, $u_{k+1} = \sum_{i=1}^{k} a_i u_i$ and so $\sum_{i=1}^{k} a_i u_i - 1u_{k+1} + \sum_{j=k+2}^{n} 0u_j = 0$ with the scalars not all zero, so S is linearly dependent.

Hence, S is linearly dependent.

2 Optional Part

1.4.1. Label the following statements as true or false.

(a) The zero vector is a linear combination of any nonempty set of vectors.

(b) The span of $\emptyset$ is $\emptyset$.

(c) If S is a subset of a vector space V, then Span(S) equals the intersection of all subspaces of V that contain S.

(d) In solving a system of linear equations, it is permissible to multiply an equation by any constant.

(e) In solving a system of linear equations, it is permissible to add any multiple of one equation to another.

(f) Every system of linear equations has a solution.

Solution:

(a) True

(b) False

(c) True

(d) False. Multiplying by zero is not permitted. Refer to MATH1050

(e) True

(f) False

1.4.4. For each list of polynomials in $\mathsf{P}_3(\mathbb{R})$, determine whether the first polynomial can be expressed as a linear combination of the other two.

(a) $x^3 - 3x + 5,\ x^3 + 2x^2 - x + 1,\ x^3 + 3x^2 - 1$

(b) $4x^3 + 2x^2 - 6,\ x^3 - 2x^2 + 4x + 1,\ 3x^3 - 6x^2 + x + 4$

(c) $-2x^3 - 11x^2 + 3x + 2,\ x^3 - 2x^2 + 3x - 1,\ 2x^3 + x^2 + 3x - 2$

(d) $x^3 + x^2 + 2x + 13,\ 2x^3 - 3x^2 + 4x + 1,\ x^3 - x^2 + 2x + 3$

(e) $x^3 - 8x^2 + 4x,\ x^3 - 2x^2 + 3x - 1,\ x^3 - 2x + 3$

(f) $6x^3 - 3x^2 + x + 2,\ x^3 - x^2 + 2x + 3,\ 2x^3 - 3x + 1$

Solution: There are multiple ways to determine the linear relation. For example, you may try to solve for the coefficients that give the desired equation and determine if such coefficients exist or not.

In the following, we shall omit the calculations and give only the final results.

(a) $x^3 - 3x + 5 = 3 \cdot (x^3 + 2x^2 - x + 1) - 2 \cdot (x^3 - 3x^2 - 1)$ is a linear combination of the other two

(b) $4x^3 + 2x^2 - 6$ is not a linear combination of the other two

(c) $-2x^3 - 11x^2 + 3x + 2 = 4 \cdot (x^3 - 2x^2 + 3x - 1) - 3 \cdot (2x^3 + x^2 + 3x - 2)$ is a linear combination of the other two

(d) $x^3 + x^2 + 2x + 13 = -2 \cdot (2x^3 - 3x^2 + 4x + 1) + 5 \cdot (x^3 - x^2 + 2x + 3)$ is a linear combination of the other two

(e) $x^3 - 8x^2 + 4x$ is not a linear combination of the other two

(f) $6x^3 - 3x^2 + x + 2$ is not a linear combination of the other two

1.4.5. In each part, determine whether the give vector is in the span of S.

(a) $(2, -1, 1),\ S = \{\ (1, 0, 2), (-1, 1, 1)\ \}$

(b) $(-1, 2, 1),\ S = \{\ (1, 0, 2), (-1, 1, 1)\ \}$

(c) $(-1, 1, 1, 2),\ S = \{\ (1, 0, 1, -1), (0, 1, 1, 1)\ \}$

(d) $(2, -1, 1, -3),\ S = \{\ (1, 0, 1, -1), (0, 1, 1, 1)\ \}$

(e) $-x^3 + 2x^2 + 3x + 3,\ S = \left\{\ x^3 + x^2 + x + 1, x^2 + x + 1, x + 1\ \right\}$

(f) $2x^3 - x^2 + x + 3,\ S = \left\{\ x^3 + x^2 + x + 1, x^2 + x + 1, x + 1\ \right\}$

(g) $\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix},\ S = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$

(h) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ S = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$

Solution: See also the note for Question 1.4.4.

(a) $(2, -1, 1) = 1 \cdot (1, 0, 2) - (-1, 1, 1)$ is in the span of S

(b) $(-1, 2, 1)$ is not in the span of S

(c) $(-1, 1, 1, 2)$ is not in the span of S

(d) $(2, -1, 1, -3) = 2 \cdot (1, 0, 1, -1) - 1 \cdot (0, 1, 1, 1)$ is in the span of S

(e) $-x^3 + 2x^2 + 3x + 3 = -1 \cdot (x^3 + x^2 + x + 1) + 3 \cdot (x^2 + x + 1) + (x + 1)$ is in the span of S

(f) $2x^3 - x^2 + x + 3$ is not in the span of S

(g) $\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + 4 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} - 2 \cdot \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is in the span of S

(h) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is not in the span of S

1.5.1. Label the following statements as true or false.

(a) If S is a linearly dependent set, then each vector in S is a linear combination of other vectors in S.

(b) Any set containing the zero vector is linearly dependent.
(c) The empty set is linearly dependent.
(d) Subsets of linearly dependent sets are linearly dependent.
(e) Subsets of linearly independent sets are linearly independent.
(f) If $a_1x_1 + \ldots + a_nx_n = 0$ and $x_1, \ldots, x_n$ are linearly independent, then all the scalars a_i are zero.

Solution:

(a) False (b) True (c) False (d) False (e) True (f) True

1.5.2. Determine whether the following sets are linearly dependent or linearly independent.

(a) $\left\{ \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} \right\}$ in $M_{2\times 2}(\mathbb{R})$
(b) $\left\{ \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & -4 \end{pmatrix} \right\}$ in $M_{2\times 2}(\mathbb{R})$
(c) $\{ x^3 + 2x^2, -x^2 + 3x + 1, x^3 - x^2 + 2x - 1 \}$ in $\mathsf{P}_3(\mathbb{R})$
(d) $\{ x^3 - x, 2x^2 - 4, -2x^3 + 3x^2 + 2x + 6 \}$ in $\mathsf{P}_3(\mathbb{R})$
(e) $\{ (1, -1, 2), (1, -2, 1), (1, 1, 4) \}$ in $\mathbb{R}^3$
(f) $\{ (1, -1, 2), (2, 0, 1), (-1, 2, -1) \}$ in $\mathbb{R}^3$
(g) $\left\{ \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ -4 & 4 \end{pmatrix} \right\}$ in $M_{2\times 2}(\mathbb{R})$
(h) $\left\{ \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 2 & -2 \end{pmatrix} \right\}$ in $M_{2\times 2}(\mathbb{R})$
(i) $\{ x^4 - x^3 + 5x^2 - 8x + 6, -x^4 + x^3 - 5x^2 + 5x - 3, x^4 + 3x^2 - 3x + 5, 2x^4 + 3x^3 + 4x^2 - x + 1, x^3 - x + 2 \}$ in $\mathsf{P}_4(\mathbb{R})$
(j) $\{ x^4 - x^3 + 5x^2 - 8x + 6, -x^4 + x^3 - 5x^2 + 5x - 3, x^4 + 3x^2 - 3x + 5, 2x^4 + x^3 + 4x^2 + 8x \}$ in $\mathsf{P}_4(\mathbb{R})$

Solution: See also the note for Question 1.4.4.

(a) $2 \cdot \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix} - \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix} = 0$, so the set is linearly dependent

(b) The set is linearly independent

(c) The set is linearly independent

(d) The set is linearly independent

(e) $3 \cdot (1, -1, 2) - 2 \cdot (2, 0, 1) - (-1, 2, -1) = 0$, so the set is linearly dependent

(f) The set is linearly independent

(g) $3 \cdot \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ -4 & 4 \end{pmatrix} = 0$, so the set is linearly dependent

(h) The set is linearly independent

(i) The set is linearly independent

(j) $4 \cdot (x^4 - x^3 + 5x^2 - 8x + 6) + 3 \cdot (-x^4 + x^3 - 5x^2 + 5x - 3) - 3 \cdot (x^4 + 3x^2 - 3x + 5) + (2x^4 + x^3 + 4x^2 + 8x) = 0$, so the set is linearly dependent

1.5.8. Let $S = \{ (1, 1, 0), (1, 0, 1), (0, 1, 1) \}$ be a subset of the vector space $\mathbb{F}^3$.

(a) Prove that if $\mathbb{F} = \mathbb{R}$, then S is linearly independent.
(b) Prove that if $\mathbb{F}$ has characteristic 2, then S is linearly independent

Solution:

(a) Let $a, b, c \in \mathbb{R}$ be such that $a \cdot (1,1,0) + b \cdot (1,0,1) + c \cdot (0,1,1) = 0 = (0,0,0)$. Then $a+b = a+b = a+c = 0$ and so $2(a+b+c) = 0$. This implies that $a+b+c = 0$, which gives $a = b = c = 0$. Hence S is linearly independent.

(b) As $\mathbb{F}$ has characteristic 2, we have $2 = 0$ and so $1 \cdot (1,1,0) + 1 \cdot (1,0,1) + 1 \cdot (0,1,1) = (2,2,2) = (0,0,0) = 0$. As the scalars are not all zero, S is linearly dependent.

Note

Recall that the characteristic of a field is the smallest $n \in \mathbb{Z}^+$ such that $\underbrace{1 + \ldots + 1}_{n\ 1s} = 0$, or 0 if no such number exists. $\mathbb{R}$ and $\mathbb{C}$ are of characteristic 0, and for every prime p there is a field $\mathbb{F}_p$ that is of characteristic p. If $\mathbb{F}$ is not of characteristic $p > 0$, then $\frac{1}{p}$ exists in $\mathbb{F}$.

1.5.10. Give an example of three linearly dependent vectors in $\mathbb{R}^3$ such that none of the three is a multiple of another.

Solution: Let $v_1 = (1,0,0), v_2 = (0,1,0), v_3 = (1,1,0) \in \mathbb{R}^3$. Then $v_1 + v_2 - v_3 = 0$, so $\{ v_1, v_2, v_2 \}$ is linearly dependent. It is easy to see that none of the vector is a multiple of the other.

1.5.16. Prove that a set S of vectors is linearly independent if and only if each finite subset of S is linearly independent.

Solution:

(a) Sppose S is linearly independent. Let $S' \subseteq S$ be a finite subset. By the Corollary of Theorem 1.6, S' is linearly independent. As S' is arbitrary, every finite subset of S is linearly independent.

(b) Suppose every finite subset of S is linearly independent. Suppose on the contrary that S is linearly dependent. Then by definition, there exists $n \in \mathbb{Z}$, $a_1, \ldots, a_n \in \mathbb{F}$, $v_1, \ldots, v_n \in S$ such that $a_1 v_1 + \ldots + a_n v_n = 0$. In particular, $S' = \{ v_1, \ldots, v_n \}$ is linearly dependent. This contradicts the assumption as S' is a finite subset of S. Hence S is linearly independent.

Therefore S is linearly independent if and only if every finite subset of S is linearly independent.

Note

Only finite sum is considered as the algebraic structure of vector space only allows finite sums. Infinite sums would require additional structures (e.g. topology).

1.5.18. Let S be a set of nonzero polynomials in $\mathsf{P}(\mathbb{F})$ such that no two have the same degree. Prove that S is linearly independent.

Solution: We will apply the result of Question 1.5.16.

Let $S' = \{ f_1, \ldots, f_n \} \subseteq S$ be a finite subset of S. Without loss of generality we may assume that $\deg f_1 < \ldots < \deg f_n$. Let $a_1, \ldots, a_n \in \mathbb{F}$ be such that $\sum_{i=1}^{n} a_i f_i = 0$, which implies that $\deg \sum_{i=1}^{n} a_i f_i = \deg 0 = -\infty$.

If some of a_i is nonzero, $\deg \sum_{i=1}^{n} a_i f_i$ is the same as $\deg f_k$ where k is the maximal index in $\{ 1, \ldots, n \}$ such that $a_k \neq 0$. By assumption, $\deg f_k \geq 0$ for all k as $f_k \in S$. So we would have $\deg \sum_{i=1}^{n} a_i f_i \geq 0$ and a contradiction would arise.

This implies that all $a_i = 0$. Hence S' is linearly independent.

Since S' is arbitrary, by the result of Question 1.5.16, S is linearly independent.

Note

Here we follow the convention that $\det 0 = -\infty$ but the choice does not matter in the argument as long as $\deg 0 < 0$.

1.5.20. Let $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ be the functions defined by $f(t) = e^{rt}$ and $g(t) = e^{st}$ where $r \neq s$. Prove that f and g are linearly independent in $\mathcal{F}(\mathbb{R}, \mathbb{R})$.

Solution: Let $a, b \in \mathbb{R}$ be such that $af + bg = 0$. Then for all $t \in \mathbb{R}$, $ae^{st} + be^{rt} = af(t) + bg(t) = 0(t) = 0$. Taking $t = 0$ and $t = 1$, we obtain $a + b = ae^s + be^r = 0$. Since $e^s \neq e^r$, we can solve the equation and obtain that $a = b = 0$. This implies that $\{ f, g \}$ is linearly independent.

Note

This can also be shown by other means, including

1. Noting that the Wronskian $W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = (s-r)e^{(r+s)t}$ is nonzero everywhere if $r \neq s$, which by classic ODE theory implies linear independence.
2. Use the result of Question 1.5.9.