MATH 2028 - Iterated integrals & Fubini's Theorem

GOAL: How to evaluate $\int_R f\,dV$ for a given bdd function $f: R \to \mathbb{R}$ which is integrable on a rectangle $R \subseteq \mathbb{R}^n$?

Recall the 1D Fundamental Theorem of Calculus:

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

provided that $f = F' : [a,b] \to \mathbb{R}$ is cts.

We will show that if the function $f: R \to \mathbb{R}$ in n variable is "nice" enough, then we can express the multiple integral $\int_R f\,dV$ as n iterated 1D integrals and the "order" of integration does not matter.

Recall: If $f = f(x,y)$ is a C^1 function on $U \subseteq \mathbb{R}^2$, then

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

More precisely, if $f : R = [a,b] \times [c,d] \to \mathbb{R}$ is cts, then we will have

$$\int_R f\,dV = \int_a^b \Big(\underbrace{\int_c^d f(x,y)\,dy}_{x \text{ fixed}} \Big)\,dx$$

$$= \int_c^d \Big(\underbrace{\int_a^b f(x,y)\,dx}_{y \text{ fixed}} \Big)\,dy$$

Fubini's Theorem will tell us exactly when the two equalities above hold. To get a feeling of how things work, let's look at a few examples.

Example 1: Consider the cts function

$$f(x,y) = xy^2$$

defined on the rectangle $R = [0,1] \times [-1,1] \subseteq \mathbb{R}^2$.

We compute the "iterated integrals" in different orders:

$$\int_{-1}^{1}\int_0^1 f(x,y)\,dx\,dy = \int_{-1}^{1} \Big(\overbrace{\int_0^1 xy^2\,dx}^{y \text{ fixed}} \Big)\,dy$$

$$= \int_{-1}^{1} y^2 \cdot \Big[\tfrac{1}{2}x^2\Big]_{x=0}^{x=1}\,dy$$

$$= \frac{1}{2}\int_{-1}^{1} y^2\,dy$$

$$= \frac{1}{2}\cdot\left[\frac{1}{3}y^3\right]_{y=-1}^{y=1} = \boxed{\frac{1}{3}}$$

SAME!

On the other hand,

$$\int_0^1\int_{-1}^1 f(x,y)\,dy\,dx = \int_0^1 \overbrace{\left(\int_{-1}^1 xy^2\,dy\right)}^{x \text{ fixed}} dx$$

$$= \int_0^1 x\cdot\left[\frac{1}{3}y^3\right]_{y=-1}^{y=1} dx$$

$$= \frac{2}{3}\int_0^1 x\,dx$$

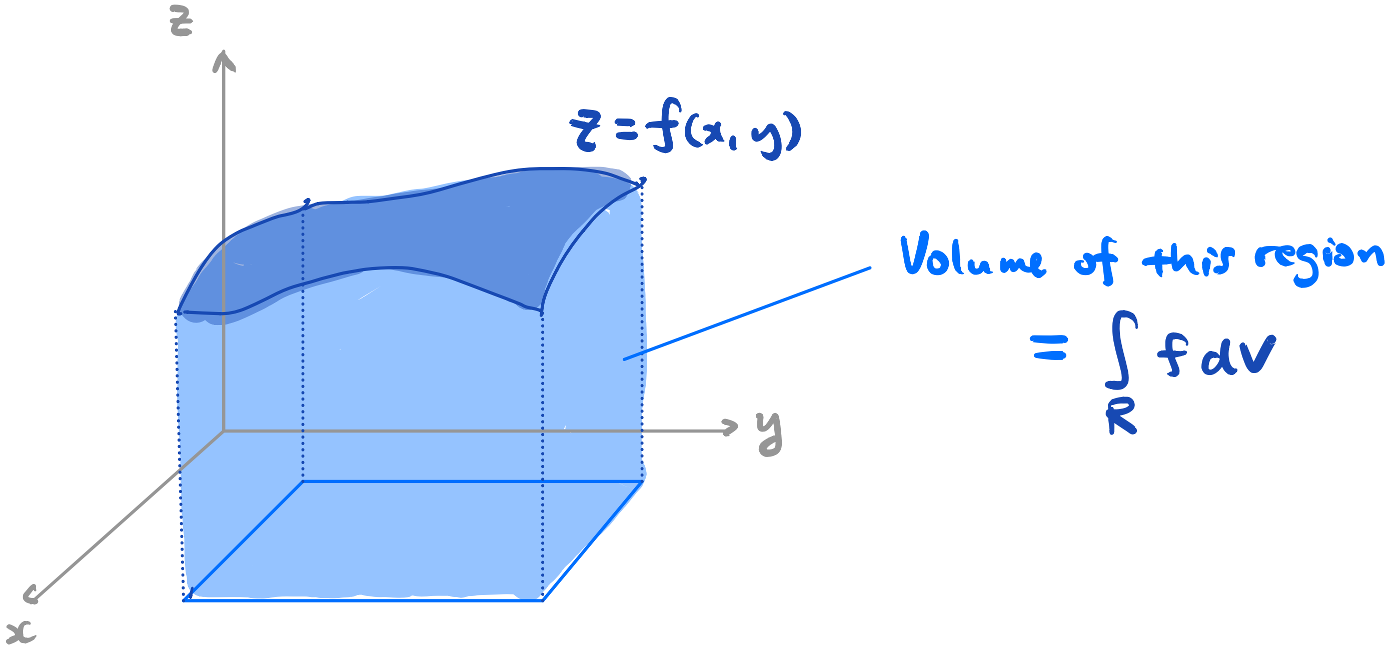
$$= \frac{2}{3}\cdot\left[\frac{1}{2}x^2\right]_{x=0}^{x=1} = \boxed{\frac{1}{3}}$$

In fact, $\int_R f\,dV = \frac{1}{3}$.

<u>Remark</u>: The underlying principle why this works is the concept of "slicing" a higher dimensional object into lower dimensional ones.

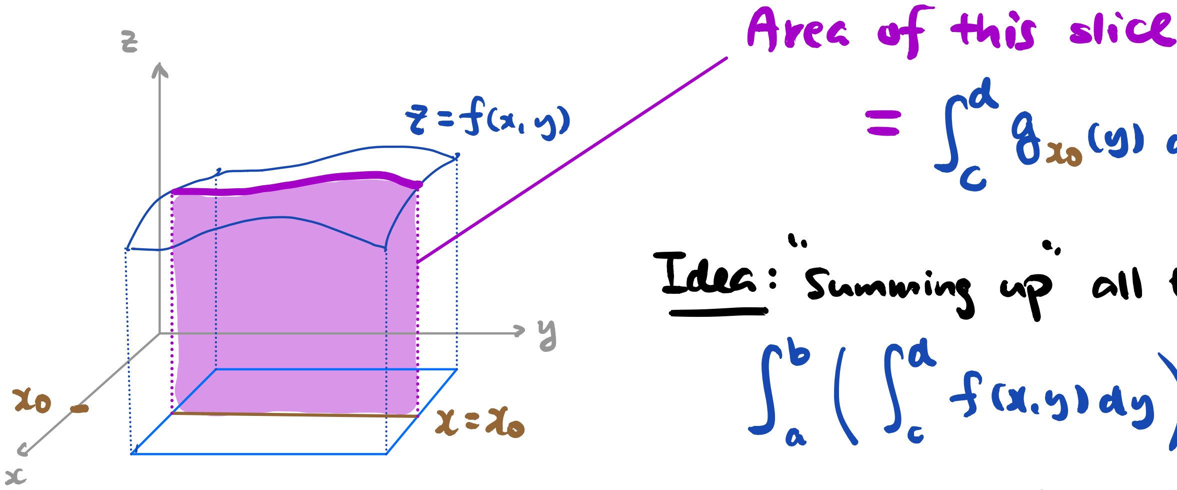
Suppose $f : R = [a,b] \times [c,d] \to \mathbb{R}$ is cts.

Recall: $\int_R f\,dV =$ Volume of the 3D region under the graph $z = f(x,y)$



Now, for EACH FIXED $x_0 \in [a,b]$, we can consider the 1-variable function $g_{x_0} : [c,d] \to \mathbb{R}$

$$g_{x_0}(y) = f(x_0, y)$$



Idea: "Summing up" all these areas

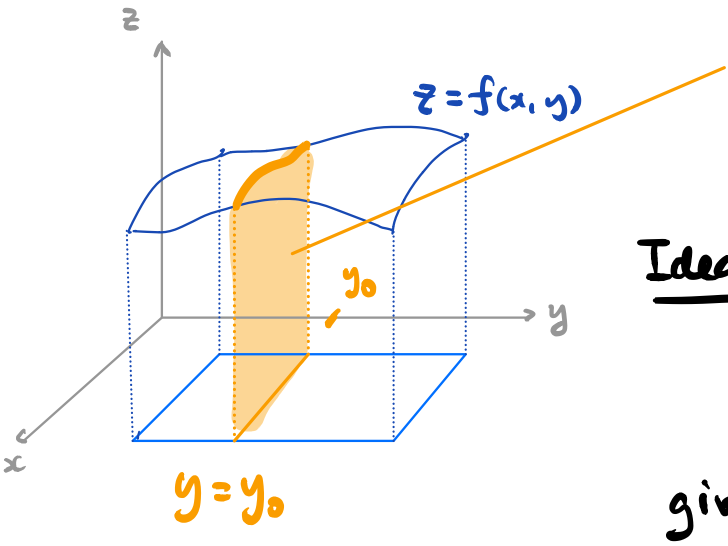
$$\int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

gives the 3D volume $\int_R f\,dV$

We can also do the "slicing" in the y-direction.

For <u>EACH FIXED</u> $y_0 \in [c,d]$, we can consider the 1-variable function $h_{y_0} : [a,b] \to \mathbb{R}$

$$h_{y_0}(x) = f(x, y_0)$$



Area of this slice

$$= \int_a^b h_{y_0}(x)\, dx$$

<u>Idea</u>: "Summing up" all these areas

$$\int_c^d \left(\int_a^b f(x,y)\, dx \right) dy$$

gives the 3D volume $\int_R f\, dV$

This heuristic idea works well <u>provided that f is "nice" enough</u> (e.g. cts). There are some subtle issues if f is only assumed to be integrable. This is shown by the following example.

Example 2: Consider $f: R = [0,1] \times [0,1] \to \mathbb{R}$,

$$f(x,y) = \begin{cases} 1 & \text{if } x=0,\ y \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

Since f vanishes identically except on the set $\{(x,y) \in R \mid x=0,\ y \in \mathbb{Q}\}$ which has measure zero, we have f is integrable and $\int_R f \, dV = 0$.

However, the function $g_0 : [0,1] \to \mathbb{R}$,

$$g_0(y) := f(0,y) = \begin{cases} 1 & \text{if } y \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

is NOT integrable (why?) on $[0,1]$ and hence $\int_0^1 g_0(y)\,dy$ does NOT exist! Nonetheless, for ANY $0 \neq x_0 \in [0,1]$, the function $g_{x_0} : [0,1] \to \mathbb{R}$ satisfies $g_{x_0}(y) := f(x_0, y) \equiv 0$, hence is integrable with $\int_0^1 g_{x_0}(y)\,dy = 0$.

To handle the situation in the example above, we introduce some terminology.

Def[n]: Let $f: R \to \mathbb{R}$ be a bdd function defined on a rectangle $R \subseteq \mathbb{R}^n$. We define the upper integral and lower integral of f on R as

$$\overline{\int_R} f\, dV := \inf_{\mathcal{P}} U(f, \mathcal{P})$$

$$\underline{\int_R} f\, dV := \sup_{\mathcal{P}} L(f, \mathcal{P})$$

Remark: Note that $\overline{\int_R} f dV$ and $\underline{\int_R} f dV$ exist regardless of whether f is integrable. Moreover, we always have

$$\underline{\int_R} f\, dV \leq \overline{\int_R} f\, dV$$

and "=" holds if and only if f is integrable on R.

Fubini's Theorem: Let $f: R \to \mathbb{R}$ be a bdd function on a rectangle $R \subseteq \mathbb{R}^n$ s.t.

$$R = A \times B$$

for some rectangles $A \subseteq \mathbb{R}^m$, $B \subseteq \mathbb{R}^k$. Denote

$$\underline{F}(x) := \underline{\int}_B f(x,y)\, dy$$

$$\overline{F}(x) := \overline{\int}_B f(x,y)\, dy$$

Notation

$f = f(x,y)$

where $x \in A$,
$y \in B$.

Suppose f is integrable on R. THEN, both $\underline{F}$ and $\overline{F}$ are integrable on A and

$$\int_R f\, dV = \int_A \underline{F}(x)\, dx = \int_A \underline{\int}_B f(x,y)\, dy\, dx$$

$$= \int_A \overline{F}(x)\, dx = \int_A \overline{\int}_B f(x,y)\, dy\, dx$$

Similarly, we have

$$\int_R f\, dV = \int_B \underline{\int}_A f(x,y)\, dx\, dy = \int_B \overline{\int}_A f(x,y)\, dx\, dy$$

Remark: The assumption that f is integrable on R is important. There exist functions whose iterated integrals exist but is **NOT** integrable.

(See Problem Set)

Proof of Fubini's Theorem:

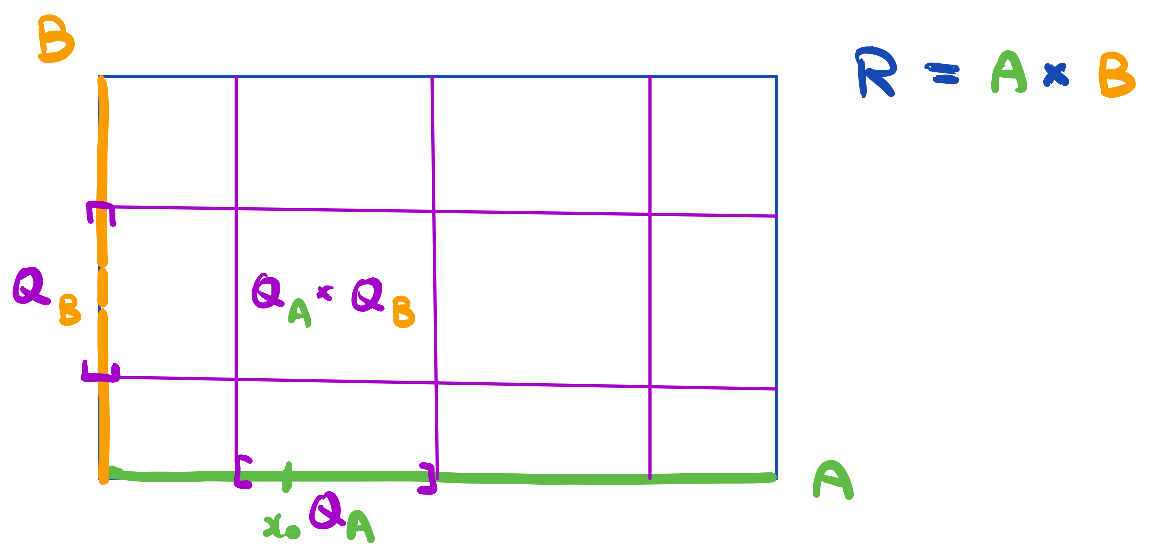
- Since $R = A \times B$, any partition $\mathcal{P}$ of R induces a partition $\mathcal{P}_A$ of A and a partition $\mathcal{P}_B$ of B s.t. if

$$\mathcal{P}_A = \{ Q_A \mid Q_A \in \mathcal{P}_A \},$$

$$\mathcal{P}_B = \{ Q_B \mid Q_B \in \mathcal{P}_B \},$$

then

$$\mathcal{P} = \{ Q_A \times Q_B \mid Q_A \in \mathcal{P}_A, Q_B \in \mathcal{P}_B \}$$



<u>Claim 1</u>: $L(f, \mathcal{P}) \le L(\underline{F}, \mathcal{P}_A)$

Take ANY $Q = Q_A \times Q_B$ where $Q_A \in \mathcal{P}_A$, $Q_B \in \mathcal{P}_B$.

Since $\inf\limits_{Q} f \le f(x_0, y) \quad \forall\, x_0 \in Q_A,\ y \in Q_B$

$$\Rightarrow \quad \inf_{Q} f \le \inf_{y \in Q_B} f(x_0, y) \quad \forall\, x_0 \in Q_A$$

Therefore, if we fixed $x_0 \in Q_A$, multiplying by $\mathrm{vol}(Q_B)$ and summing over all $Q_B \in \mathcal{P}_B$.

$$\sum_{Q_B} \inf_{Q_A \times Q_B} f \cdot \mathrm{vol}(Q_B)$$

$$\le \sum_{Q_B} \inf_{y \in Q_B} f(x_0, y) \cdot \mathrm{vol}(Q_B)$$

a function of y since x_0 is fixed

$$= L(\boxed{f(x_0, y)}, \mathcal{P}_B)$$

$$\le \int_{\underline{B}} f(x_0, y)\, dy =: \underline{F}(x_0)$$

Since the above inequalities hold for **EACH** fixed $x_0 \in Q_A$, we have

$$\sum_{Q_B} \inf_{Q_A \times Q_B} f \cdot \mathrm{vol}(Q_B) \leq \inf_{x \in Q_A} \underline{F}(x)$$

Multiply by $\mathrm{Vol}(Q_A)$ and sum over all $Q_A \in \mathcal{P}_A$.

$$\begin{aligned} L(f, \mathcal{P}) &= \sum_{Q} \inf_{Q} f \cdot \mathrm{Vol}(Q) \\ &= \sum_{Q_A} \sum_{Q_B} \inf_{Q_A \times Q_B} f \cdot \mathrm{vol}(Q_B) \cdot \mathrm{vol}(Q_A) \\ &\leq \sum_{Q_A} \inf_{Q_A} \underline{F} \cdot \mathrm{Vol}(Q_A) = L(\underline{F}, \mathcal{P}_A) \end{aligned}$$

which proves the claim.

Claim 2: $U(\overline{F}, \mathcal{P}_A) \leq U(f, \mathcal{P})$

The proof is similar to Claim 1 and hence left as an exercise.

In summary, we have the following relations

$$
\begin{tikzcd}
 & & U(\underline{F},\mathcal{P}_A) \arrow[dr, phantom, "\leq \ \because \underline{F}\leq\overline{F}"] & & \\
L(f,\mathcal{P}) \arrow[r, phantom, "\overset{\text{Claim 1}}{\leq}"] & L(\underline{F},\mathcal{P}_A) \arrow[ur, phantom, "\leq \ \text{trivial}"] \arrow[dr, phantom, "\because \underline{F}\leq\overline{F} \ \leq"] & & U(\overline{F},\mathcal{P}_A) \arrow[r, phantom, "\overset{\text{Claim 2}}{\leq}"] & U(f,\mathcal{P}) \\
 & & L(\overline{F},\mathcal{P}_A) \arrow[ur, phantom, "\leq \ \text{trivial}"] & &
\end{tikzcd}
$$

<u>Claim 3</u>: $\underline{F}$, $\overline{F}$ are integrable over A.

Since f is integrable over R by assumption, we have from Riemann condition that $\forall\, \varepsilon > 0$, $\exists$ partition $\mathcal{P}$ of R s.t.

$$U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon$$

By the diagram above, we have

$$U(\underline{F},\mathcal{P}_A) - L(\underline{F},\mathcal{P}_A) < \varepsilon$$

$$U(\overline{F},\mathcal{P}_A) - L(\overline{F},\mathcal{P}_A) < \varepsilon$$

This proves the claim by Riemann condition again.

Finally, using the diagram again, we have

$$\int_R f\,dV = \int_A \underline{F}\,dV = \int_A \overline{F}\,dV$$

$\blacksquare$

Example 2 (revisited):

Consider $f : R = [0,1] \times [0,1] \to \mathbb{R}$,

$$f(x,y) = \begin{cases} 1 & \text{if } x = 0,\ y \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

One checks that

$$\underline{F}(x) = \underline{\int_0^1} f(x,y)\,dy = 0 \qquad \forall x \in [0,1]$$

$$\overline{F}(x) = \overline{\int_0^1} f(x,y)\,dy = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

and

$$\int_R f\,dV = \int_0^1 \underline{F}(x)\,dx = \overline{\int_0^1} \overline{F}(x)\,dx = 0$$

Example 3: Let $f : R = [0,1] \times [0,1] \to \mathbb{R}$ s.t.

$$f(x,y) = \begin{cases} 1 - \frac{1}{q}, & \text{if } y \in \mathbb{Q},\ x = \frac{p}{q} \in \mathbb{Q}_{>0} \\ & \text{where } p, q \in \mathbb{N} \text{ are coprime} \\ 1, & \text{otherwise} \end{cases}$$

Note that f is integrable on R with $\int_R f\,dV = 1$.

(Verify this!) On the other hand,

$$\underline{F}(x) = \underline{\int_0^1} f(x,y)\,dy = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \in \mathbb{Q}_{>0} \\ 1 & \text{otherwise} \end{cases}$$

$$\overline{F}(x) = \overline{\int_0^1} f(x,y)\,dy = 1 \qquad \forall x \in [0,1]$$

Therefore, we have

$$1 = \int_R f\,dV = \int_0^1 \underline{F}(x)\,dx = \int_0^1 \overline{F}(x)\,dx$$