P.70 : 1, 3

1. Let $z = x + iy$

(a) $f(z) = \bar{z} = x - iy$, $u(x,y) = x$, $v(x,y) = -y$

4'

$\Rightarrow u_x = 1,\ u_y = 0,\ v_x = 0,\ v_y = -1$

$\Rightarrow u_x \neq v_y$ for $\forall x, y \in \mathbb{R}$ $\Rightarrow$ not satisfy CR equations.

$\Rightarrow$ $f'(z)$ DNE

(b) $f(z) = z - \bar{z} = i2y$, $u(x,y) = 0$, $v(x,y) = 2y$

4'

$\Rightarrow u_x = 0,\ u_y = 0,\ v_x = 0,\ v_y = 2$

$\Rightarrow u_x \neq v_y$ for $\forall x, y \in \mathbb{R}$ $\Rightarrow$ not satisfy CR equations.

$\Rightarrow$ $f'(z)$ DNE

(c) $f(z) = 2x + ixy^2$, $u(x,y) = 2x$, $v(x,y) = xy^2$

4'

$\Rightarrow u_x = 2,\ u_y = 0,\ v_x = y^2,\ v_y = 2xy$

if $u_x = v_y$, $u_y = -v_x$, then

$$\begin{cases} 2 = 2xy \\ 0 = -y^2 \end{cases}$$

, such x, y does not exist in $\mathbb{R}$

$\Rightarrow$ not satisfy CR equations, $f'(z)$ DNE.

(d) $f(z) = e^x e^{-iy} = e^x \cos y - i e^x \sin y$,

4'

$u(x,y) = e^x \cos y$, $v(x,y) = -e^x \sin y$

$u_x = e^x \cos y$, $u_y = -e^x \sin y$, $v_x = -e^x \sin y$, $v_y = -e^x \cos y$

if $u_x = v_y$, $u_y = -v_x$, then

integer.

$e^x \cos y = -e^x \cos y \Rightarrow y = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

$-e^x \sin y = e^x \sin y \Rightarrow y = k\pi,\ k \in \mathbb{Z}$

$\Rightarrow$ such y does not exist.

$\Rightarrow$ not satisfy CR equations, $f'(z)$ DNE.

3. Let $z = x + iy$

(a) $f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$.

3'

$u(x,y) = \frac{x}{x^2+y^2}$, $v(x,y) = \frac{-y}{x^2+y^2}$

$u_x = \frac{y^2-x^2}{(x^2+y^2)^2}$, $u_y = \frac{-2xy}{(x^2+y^2)^2}$, $v_x = \frac{2xy}{(x^2+y^2)^2}$, $v_y = \frac{y^2-x^2}{(x^2+y^2)^2}$

define on $x^2+y^2 \neq 0$, i.e. $z \neq 0$.

continuous on $\mathbb{R}^2 \setminus \{0\}$, and

$u_x = v_y$, $u_y = -v_x$

$\Rightarrow$ $f'(z) = u_x + iv_x = \frac{y^2-x^2+i\cdot 2xy}{(x^2+y^2)^2} = \frac{-(\bar{z})^2}{(z\bar{z})^2} = -\frac{1}{z^2}$ $(z \neq 0)$

(b) $f(z) = x^2 + iy^2$, $u(x,y) = x^2$, $v(x,y) = y^2$

3'

$u_x = 2x$, $u_y = 0$, $v_x = 0$, $v_y = 2y$

Let $u_x = v_y$. $2x = 2y$ $\Rightarrow$ $x = y$

$u_y = -v_x = 0$

continuous on $\{(x,y) \in \mathbb{R}^2 : x = y\}$

$\Rightarrow$ $f'(z)$ exists on $\{z = x+iy \in \mathbb{C} : x = y\}$.

$\Rightarrow$ $f'(x+ix) = u_x + iv_x = 2x$.

(c) $f(z) = z\,\mathrm{Im}\,z = (x+iy)\cdot y = xy + iy^2$, $u(x,y) = xy$, $v(x,y) = y^2$

3'

$u_x = y$, $u_y = x$, $v_x = 0$, $v_y = 2y$

Let $u_x = v_y$. $y = 2y$ $\Rightarrow$ $y = 0$

$u_y = -v_x$, $\Rightarrow$ $x = 0$

continuous on $\{(x,y) \in \mathbb{R}^2 : x = y = 0\}$

$\Rightarrow$ $f'(z)$ exists on $z = 0$

$\Rightarrow$ $f'(0) = u_x + iv_x = 0$.

P. 76.

7. $f(z) = u(x,y) + i v(x,y)$.
5'
$= u(x,y)$ since f is real-valued.

Given f is analytic. by CR eq'ns.

$u_x = v_y = 0$, $u_y = -v_x = 0$

$\Rightarrow$ $u(x,y) = C$ is a constant function.

$\Rightarrow$ $f(z)$ is constant.

P. 89

3. Let $z = x + iy$,
5'
$f(z) = \exp \bar{z} = e^{x-iy} = e^x \cos y - i \cdot e^x \sin y$

$u(x,y) = e^x \cos y$, $v(x,y) = -e^x \sin y$

$u_x = e^x \cos y$, $u_y = -e^x \sin y$, $v_x = -e^x \sin y$, $v_y = -e^x \cos y$

if $u_x = v_y$, $u_y = -v_x$, then

$$\left.\begin{array}{l} e^x \cos y = -e^x \cos y \Rightarrow y = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z} \\ -e^x \sin y = e^x \sin y \Rightarrow y = k\pi,\ k \in \mathbb{Z} \end{array}\right\} \Rightarrow \text{such } y \text{ does not exist.}$$

$\Rightarrow$ not satisfy CR equations, $f'(z)$ DNE.

10. (a) Let $z = x + iy$
5'
$e^z = e^x \cos y + i e^x \sin y = e^x \cos y$ since e^z is real.

$\Rightarrow$ $\sin y = 0$ $\Rightarrow$ $y = \operatorname{Im} z = n\pi$ $(n = 0, \pm 1, \pm 2, \cdots)$.

since $e^x > 0$ for $\forall x \in \mathbb{R}$.

5'
(b). by (a), $e^x \cos y = 0$, $e^x > 0 \quad \forall x \in \mathbb{R}$.

$\Rightarrow \cos y = 0 \Rightarrow y = \frac{\pi}{2} + n\pi$, $(n = 0, \pm 1, \pm 2, \cdots)$.

$\Rightarrow \operatorname{Im} z = \frac{\pi}{2} + n\pi$, $(n = 0, \pm 1, \pm 2, \cdots)$

P.96.

8. $z = e^{\log z} = e^{\frac{i\pi}{2}} = i$

5'

10. (a) $z - i \neq 0 \Rightarrow z \neq i$

5' since $\log z$ is undefined at $z = 0$.

Let $z = x + iy$,

$\operatorname{Log}(z)$ is analytic for $|z| > 0$, $-\pi < \operatorname{Arg} z < \pi$.

$\Rightarrow \operatorname{Arg}(z - i) = \operatorname{Arg}(x + i(y-1)) = \pi$. (branch cut).

$\Rightarrow x < 0$ and $y = 1$

Combine with $z \neq i$, then $f(z)$ is analytic

except on the portion $x \leq 0$ of $y = 1$.

(b) $z^2 + i = 0 \Rightarrow (x + iy)^2 + i = 0$

5' $\Rightarrow x^2 - y^2 + i(1 + 2xy) = 0$

$\Rightarrow x^2 = y^2$, $1 + 2xy = 0$

$\Rightarrow x + iy = \frac{1-i}{\sqrt{2}}$ or $x + iy = \frac{-1+i}{\sqrt{2}}$

$z + 4 = 0 \Rightarrow z = -4$.

$\operatorname{Arg}(z+4) = \operatorname{Arg}(x + 4 + iy) = \pi$ (branch cut).

$\Rightarrow x + 4 < 0$, $y = 0$

$\Rightarrow f(z)$ analytic except at $\pm\frac{1-i}{\sqrt{2}}$ and $x \leq -4$ of real axis.

P. 103.

1. (a) $(1+i)^i = e^{i\log(1+i)} = e^{i\left(\ln\sqrt{2}+\left(\frac{\pi}{4}+2k\pi\right)i\right)}$, $k=0,\pm1,\pm2,\cdots$

5'

$\overset{n=-k}{=} \exp\left(-\frac{\pi}{4}+2n\pi\right)\exp\left(i\frac{\ln 2}{2}\right)$, $n=0,\pm1,\cdots$

(b) $\frac{1}{i^{2i}} = i^{-2i} = e^{-2i\log i} = e^{-2i\left(\left(\frac{\pi}{2}+2n\pi\right)i\right)}$, $n=0,\pm1,\pm2,\cdots$

5'

$\overset{n=-k}{=} \exp(\pi+4n\pi)$

$= \exp[(4n+1)\pi]$ $n=0,\pm1,\pm2,\cdots$

6. P.V. $z^a = e^{a\operatorname{Log} z} = e^{a(\ln|z|+i\operatorname{Arg} z)}$, $|z|>0$, $-\pi < \operatorname{Arg} z \leq \pi$.

5'

$\Rightarrow |z^a| = |e^{a\ln|z|}\cdot e^{ia\operatorname{Arg} z}|$

$= |e^{a\ln|z|}|\cdot|e^{ia\operatorname{Arg} z}|$

$= |z|^a$

since $|e^{i\theta}| = 1$, $\forall\, \theta\in\mathbb{R}$.

Note that: $\operatorname{Log} z$ is defined on $\{|z|>0,\ -\pi<\operatorname{Arg} z\leq\pi\}$ but only analytic on $\{|z|>0,\ -\pi<\operatorname{Arg} z<\pi\}$.

Not domain of $\operatorname{Log} z$.

In other words, $\operatorname{Log} z$ is defined on negative part of real axis, but not analytic or even continuous on it.

9. $c^{f(z)} = \exp(f(z)\log c)$

5'

$\Rightarrow \frac{d}{dz}c^{f(z)} = \exp(f(z)\log c)\cdot\log c\cdot f'(z)$

$= c^{f(z)}\cdot\log c\cdot f'(z)$

P. 108

8. (a) $|\sin z|^2 = \sin^2 x + \sinh^2 y \overset{\sinh^2 y\geq 0}{\geq} \sin^2 x \geq 0$

$\Rightarrow |\sin z| \geq |\sin x|$

5'

(b) $|\cos z|^2 \geq \cos^2 x + \sinh^2 y \overset{\sinh^2 y\geq 0}{\geq} \cos^2 x \geq 0$

$\Rightarrow |\cos z| \geq |\cos x|$.

9. (a) $\sinh^2 y = |\sin z|^2 - \sin^2 x \le |\sin z|^2$

5' $|\sin z|^2 = \sin^2 x + \cosh^2 y - 1$

$\Rightarrow \cosh^2 y = |\sin z|^2 + 1 - \sin^2 x$

$= |\sin z|^2 + \cos^2 x$

$\geqslant |\sin z|^2$

$\Rightarrow \underline{|\sinh y| \le |\sin z| \le \cosh y}$

5' (b) $\sinh^2 y = |\cos z|^2 - \cos^2 x \le |\cos z|^2$

$|\cos z|^2 = \cos^2 x + (\cosh^2 y - 1)$

$\Rightarrow \cosh^2 y = |\cos z|^2 + 1 - \cos^2 x$

$= |\cos z|^2 + \sin^2 x$

$\geqslant |\cos z|^2$.

$\Rightarrow \underline{\sinh y \le |\cos z| \le \cosh y}$.

P114

3. $\cos z = \sqrt{2}$

5' $\Rightarrow \dfrac{e^{iz} + e^{-iz}}{2} = \sqrt{2}$

$u = e^{iz}$ $\Rightarrow \dfrac{u + u^{-1}}{2} = \sqrt{2}$

$\Rightarrow u^2 - 2\sqrt{2}u + 1 = 0$

$\Rightarrow u = \sqrt{2} + 1$ or $u = \sqrt{2} - 1$

① $e^{iz} = u = \sqrt{2} + 1$

$\Rightarrow z = -i \log(\sqrt{2} + 1) + 2k\pi$

$k = 0, \pm 1, \pm 2, \cdots$

② $e^{iz} = u = \sqrt{2} - 1$

$\Rightarrow z = -i \log(\sqrt{2} - 1) + 2k\pi$

$\Rightarrow \underline{z = -i \ln(\sqrt{2} \pm 1) + 2k\pi \overset{\text{OR}}{=} i \ln(\sqrt{2} \mp 1) + 2k\pi. \ (k = 0, \pm 1, \pm 2, \cdots)}$