P13.5 (3×3) HW1

(a) $|z-1+i| = |z-(1-i)| = 1.$

The set of points forms a circle centered at $1-i$ with radius 1.

(b) $|z+i| = |z-(-i)| \leq 3$

The set of points forms a closed circle disc centered at $-i$ with radius 3.

(c) $|z-4i| \geq 4$

The set of points forms the complement of an open circle disc centered at $+4i$ with radius 4.

7. (7) $p(z) = a_0 + a_1 z + \cdots a_n z^n$

$$= z^n \left(\frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots \frac{a_{n-1}}{z} + a_n \right) \quad (*)$$

We write $w = \frac{a_0}{z^n} + \cdots + \frac{a_{n-1}}{z}.$

$$|w| \leq \frac{|a_0|}{|z|^n} + \cdots + \frac{|a_{n-1}|}{|z|}$$

We can find a sufficiently large positive number R s.t $|w| < |a_n|$ when $|z| > R$.

In view of equation $(*)$, we have

$$|p(z)| = |z^n(w+a_n)| \leq |z|^n(|w| + |a_n|)$$
$$< 2|a_n||z|^n, \text{ when } |z| > R.$$

(3) P16.7.

$$|Re(2+\bar{z}+z^3)| \leq |2+\bar{z}+z^3|$$
$$\leq 2 + |\bar{z}| + |z|^3$$
$$\leq 2 + |z| + |z|^3$$
$$\leq 2+1+1 = 4 \quad (|z| \leq 1 \text{ used})$$

P24.5 (4×3)

(a) $i = e^{i\frac{\pi}{2}}$

$1-\sqrt{3}i = 2e^{-i\frac{\pi}{3}}$

$\sqrt{3}+i = 2e^{i\frac{\pi}{6}}$

$$i(1-\sqrt{3}i)(\sqrt{3}+i) = 4e^{i\left(\frac{\pi}{2}-\frac{\pi}{3}+\frac{\pi}{6}\right)}$$
$$= 4e^{i\frac{\pi}{3}}$$
$$= 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$
$$= 4\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 2 + 2\sqrt{3}i$$

(b) $i = e^{i\frac{\pi}{2}}$

$2+i = \sqrt{5}e^{i\theta} \left(\cos\theta = \frac{2}{\sqrt{5}}, \sin\theta = \frac{1}{\sqrt{5}}\right)$

$$\frac{5i}{2+i} = \frac{5e^{i\frac{\pi}{2}}}{\sqrt{5}e^{i\theta}} = \sqrt{5}e^{i\left(\frac{\pi}{2}-\theta\right)}$$
$$= \sqrt{5}\left(\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)\right)$$
$$= \sqrt{5}(\sin\theta + i\cos\theta)$$
$$= \sqrt{5}\left(\frac{1}{\sqrt{5}} + i\cdot\frac{2}{\sqrt{5}}\right) = 1+2i$$

(c) $\sqrt{3}+i = 2e^{i\frac{\pi}{6}}$

$(\sqrt{3}+i)^6 = 64e^{i\pi} = -64$

(d) $1+\sqrt{3}i = 2e^{i\frac{\pi}{3}}$

$$(1+\sqrt{3}i)^{-10} = 2^{-10}e^{-i\frac{10\pi}{3}} = 2^{-10}\cdot e^{i\frac{2\pi}{3}}$$
$$= 2^{-10}\left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right)$$
$$= 2^{-10}\left(\frac{-1}{2} + i\frac{\sqrt{3}}{2}\right)$$
$$= 2^{-11}(-1+\sqrt{3}i)$$

(7) 9. Let $S = 1 + z + z^2 + \cdots + z^n$, and so

$$zS = z + z^2 + \cdots z^n + z^{n+1}.$$

$$S - zS = 1 - z^{n+1} \Rightarrow S = \frac{1-z^{n+1}}{1-z} \quad (z \neq 1)$$

Take $z = e^{i\theta}$ in the above equality. We have

$$1 + \cos\theta + i\sin\theta + \cos 2\theta + i\sin 2\theta + \cdots \cos n\theta + i\sin n\theta$$
$$= \frac{1-e^{(n+1)i\theta}}{1-e^{i\theta}},$$ here we use $e^{in\theta} = \cos n\theta + i\sin n\theta$

The real part of the left-hand side is

$1 + \cos\theta + \cos 2\theta + \cdots \cos n\theta,$

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The right-hand side $= \frac{1-e^{(n+1)i\theta}}{1-e^{i\theta}}$

$$= \frac{e^{-i\theta/2}-e^{in\theta+i\theta/2}}{e^{-i\theta/2}-e^{i\theta/2}}$$

$$= \frac{\cos\frac{\theta}{2}-i\sin\frac{\theta}{2}-\cos(n\theta+\frac{\theta}{2})-i\sin(n\theta+\frac{\theta}{2})}{-2i\sin\frac{\theta}{2}}$$

$$= \frac{i\cos\frac{\theta}{2}+\sin\frac{\theta}{2}-i\cos(n\theta+\frac{\theta}{2})+\sin(n\theta+\frac{\theta}{2})}{2\sin\frac{\theta}{2}}$$

It is obvious that the real part of the right-hand side is $\frac{\sin\frac{\theta}{2}+\sin(n\theta+\frac{\theta}{2})}{2\sin\frac{\theta}{2}}$

The real part of the left-hand side should be equal to that of the right-hand side, so $1+\cos\theta+\cdots\cos n\theta = \frac{1}{2}+\frac{\sin[(2n+1)\theta/2]}{2\sin\theta/2}$.

P31.3 (7) $-8-8\sqrt{3}i = +8(-1-\sqrt{3}i) = +8\cdot 2\cdot e^{-i\frac{2\pi}{3}} = +16e^{-i\frac{2\pi}{3}}$

$\Rightarrow r_0=16,\ \theta_0=-\frac{2\pi}{3},\ -8-8\sqrt{3}i = r_0e^{i\theta_0}$

If $(-8-8\sqrt{3}i)^{1/4} = re^{i\theta}$, we have

$r_0e^{i\theta_0} = r^4e^{i4\theta}$

$\Rightarrow r=r_0^{1/4},\ \theta_0=4\theta+2k\pi,\ k=0,\pm1,\cdots$

$\Rightarrow r=r_0^{1/4}=2,$

$\theta=\frac{\theta_0}{4}+\frac{2k\pi}{4},\ k=0,\pm1,\cdots$

$\Rightarrow c_k=2\exp(-i\frac{\pi}{6}+\frac{ik\pi}{2}),\ k=0,1,2,3.$

$\Rightarrow c_0=2e^{i(-\frac{\pi}{6})}=2(\frac{\sqrt{3}}{2}-\frac{1}{2}i)=\sqrt{3}-i$

$c_1=2\cdot e^{i(-\frac{\pi}{6}+\frac{\pi}{2})}=2(\frac{1}{2}+\frac{\sqrt{3}}{2}i)=1+\sqrt{3}i$

$c_2=2\cdot e^{i(-\frac{\pi}{6}+\pi)}=2(-\frac{\sqrt{3}}{2}+\frac{1}{2}i)=-\sqrt{3}+i$

$c_3=2\cdot e^{i(-\frac{\pi}{6}+\frac{3}{2}\pi)}=2(-\frac{1}{2}-\frac{\sqrt{3}}{2}i)=-1-\sqrt{3}i$

$\Rightarrow c_k=c_0e^{i(\frac{\pi}{2}k)},\ k=0,1,2,3$

c_0 is the principle root, since $e^{i\frac{\pi k}{2}}$ represents a counterclockwise rotation through $\frac{\pi k}{2}$ radians and c_k has the same module, roots are the vertices of a certain square.

7. (7) From Ex 9, sec 9, we know that

$$1+z+z^2+\cdots+z^n=\frac{1-z^{n+1}}{1-z},\ z\neq 1$$

If $c^n=1$ and $c\neq 1$, then

$$1+c+c^2+\cdots+c^{n-1}+1=\frac{1-c^{n+1}}{1-c}=\frac{1-c}{1-c}=1.$$

Therefore, $1+c+c^2+\cdots c^{n-1}=0.$

P43.4 (3) $z=re^{i\theta}$, $f(z)=re^{i\theta}+r^{-1}e^{-i\theta}$

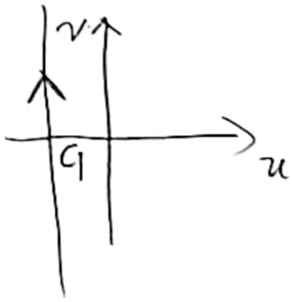
$$=r(\cos\theta+i\sin\theta)+r^{-1}(\cos\theta-i\sin\theta)$$

$$=(r+\frac{1}{r})\cos\theta+i(r-\frac{1}{r})\sin\theta$$

6. (16) $z=x+iy$, $w=(x+iy)(x+iy)$

$$=x^2-y^2+2xyi$$

① $x^2-y^2=c_1<0$



[1] upper branch

$y=\sqrt{x^2-c_1}$

$w=c_1+2x\sqrt{x^2-c_1}\,i$, $x\in(-\infty,+\infty)$

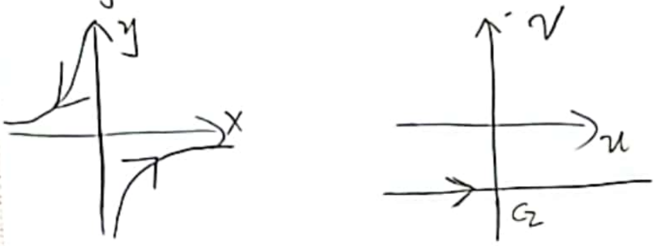
The image of a point (x,y) on that branch moves upward along the entire line as (x,y) traces out the branch towards the right.

[2] lower branch

$y=-\sqrt{x^2-c_1}$, $w=c_1-2x\sqrt{x^2-c_1}\,i$, $x\in(-\infty,+\infty)$

The image of ...... as (x,y) traces out the branch towards the left.

② $2xy = c_2 < 0$



[1] branch lying in the second quadrant.

$y = \frac{c_2}{2x}$

$w = x^2 - \frac{c_2^2}{4x^2} + c_2 i \quad (-\infty < x < 0)$

$\lim_{x\to 0^-} x^2 - \frac{c_2^2}{4x^2} = -\infty, \quad \lim_{x\to\infty} x^2 - \frac{c_2^2}{4x^2} = \infty.$

It is clear that as (x,y) travels down the entire branch, its image moves to the right along the entire line $v=c_2$.

[2] branch lying in the fourth quadrant.

$y = \frac{c_2}{2x}$

$w = x^2 - \frac{c_2^2}{4x^2} + c_2 i, \quad (0 < x < \infty)$

$\lim_{x\to 0^+} x^2 - \frac{c_2^2}{4x^2} = -\infty, \quad \lim_{x\to\infty} x^2 - \frac{c_2^2}{4x^2} = \infty.$

As (x,y) moves up along the branch, its image moves towards the right along the entire line $v=c_2$.

P55. 10, 3x3

(a) $\lim_{z\to\infty} \frac{4z^2}{(z-1)^2} = \lim_{z\to 0} \frac{4/z^2}{(\frac{1}{z}-1)^2}$

$= \lim_{z\to 0} \frac{4}{(1-z)^2} = 4$

(b) $\lim_{z\to 1} (z-1)^3 = 0$

$\Rightarrow \lim_{z\to 1} \frac{1}{(z-1)^3} = \infty$

(c) $\lim_{z\to\infty} \frac{z-1}{z^2+1} = \lim_{z\to 0} \frac{1/z - 1}{1/z^2 + 1}$

$= \lim_{z\to 0} \frac{z-z^2}{1+z^2} = 0$

$\Rightarrow \lim_{z\to\infty} \frac{z^2+1}{z-1} = \infty$

11. 2x4

(a) $T(z) = \frac{az+b}{d}$ if $c=0$

$\lim_{z\to\infty} T(z)^{-1} = \lim_{z\to\infty} \frac{d}{az+b} = \lim_{z\to 0} \frac{d}{a/z+b}$

$= \lim_{z\to 0} \frac{dz}{a+bz} = 0$

$\Rightarrow \lim_{z\to\infty} T(z) = \infty$

(b) $\lim_{z\to\infty} T(z) = \lim_{z\to 0} \frac{a/z+b}{c/z+d}$

$= \lim_{z\to 0} \frac{a+bz}{c+dz} = \frac{a}{c}$

$\lim_{z\to -\frac{d}{c}} T(z)^{-1} = \lim_{z\to -\frac{d}{c}} \frac{cz+d}{az+b} = 0$

$\Rightarrow \lim_{z\to -\frac{d}{c}} T(z) = \infty$

P61. 4 5

$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{f(z)-f(z_0)/z-z_0}{g(z)-g(z_0)/z-z_0}$

$= \frac{\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}}{\lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0}}$

$= \frac{f'(z_0)}{g'(z_0)}$

7 9. $\frac{\Delta w}{\Delta z} = \frac{f(z+\Delta z)-f(z)}{\Delta z}, \quad \Delta z = \Delta x + i\Delta y.$

when $z=0$, $\frac{\Delta w}{\Delta z} = \frac{f(\Delta z)}{\Delta z} = \frac{\overline{\Delta z}^2}{\Delta z^2} = \frac{(\Delta x - i\Delta y)^2}{(\Delta x + i\Delta y)^2}$

① $\Delta x = 0$, $\frac{\Delta w}{\Delta z} = \frac{-\Delta y^2}{-\Delta y^2} = 1.$

② $\Delta y = 0$, $\frac{\Delta w}{\Delta z} = \frac{\Delta x^2}{\Delta x^2} = 1$

③ $\Delta y = \Delta x$, $\frac{\Delta w}{\Delta z} = \frac{-2i\Delta x^2}{2i\Delta x^2} = -1$

If $f'(0)$ exists, $f'(0) = \lim_{\Delta z\to 0} \frac{\Delta w}{\Delta z} = \lim_{(\Delta x,\Delta y)\to 0} \frac{\Delta w}{\Delta z}$

$= \lim_{(\Delta x,0)\to 0} \frac{\Delta w}{\Delta z} = 1$

$\neq \lim_{(\Delta x,\Delta x)\to 0} \frac{\Delta w}{\Delta z} = -1.$ 3

This is a contradiction.