Recall

Fundamental theorems

Open Mapping Theorem Let X, Y be **Banach** spaces and $T \in B(X, Y)$ be **surjective**. Then T is a open mapping.

- If $T \in B(X, Y)$ is a bijection, then T^{-1} is bounded.
- Denote the image of an operator A by $\operatorname{Im}(A)$. Let $T, K \in B(X, Y)$. Then $\{\ \operatorname{Im}(T)$ is closed and $\dim \operatorname{Im}(K) < \infty\ \} \implies \{\ \operatorname{Im}(T + K)$ is closed $\}$. However, if we weaken $\dim \operatorname{Im}(K) < \infty$ to $\operatorname{Im}(K)$ being closed, then "$\implies$" may not hold.

Closed Graph Theorem Let X, Y be **Banach** spaces and $T : X \to Y$ be a linear operator. Then $\{\ T$ is bounded $\} \iff \{\ T$ has closed graph $\}$.

- A general approach to prove the boundedness of a linear map $T : X \to Y$: Suppose $x_n \to x \in X$ and $T_n x \to y \in Y$. If we can check $Tx = y$, then T is continuous.

Uniform Boundedness Theorem Let X be a **Banach** space and Y be a normed space. Let $(T_i)_{i\in I}$ be a family of bounded linear operators from X to Y. Suppose for all $x \in X$, we have $\sup_{i\in I}\|T_i(x)\| < \infty$. Then $\sup_{i\in I}\|T_i\| < \infty$.

- Let $(T_n)_{n=1}^{\infty} \in B(X, Y)$. Suppose $\lim_{n\to\infty} T_n(x)$ exists in Y for all $x \in X$. Then there exists $T \in B(X, Y)$ such that $T(x) = \lim_{n\to\infty} T_n(x)$ for all $x \in X$ and $\|T\| \leq \liminf_{n\to\infty}\|T_n\|$.
- Weakly convergent sequences in normed spaces are bounded.

To prove the above theorems, it is essential to exploit the completeness of Banach spaces via *Baire Category Theorem*.

Dual spaces of subspaces and quotient spaces

Let X, Y be Banach spaces and M be a closed subspace of X. For convenience we first introduce a symmetric notation. For $x \in X$ and $x^* \in X$, denote

$$\langle x, x^*\rangle := x^*(x).$$

Then let $T \in B(X, Y)$. The *adjoint operator* $T^* : Y^* \to X^*$ is defined by

$$\langle x, T^* y^*\rangle := \langle Tx, y^*\rangle \quad \forall\, y^* \in Y^*, \forall\, x \in X \tag{1}$$

with $\|T^*\| = \|T\|$. The *canonical map* $Q : X \to X^{**}$ is defined by

$$\langle x^*, Qx\rangle := \langle x, x^*\rangle \quad \forall\, x \in X, \forall\, x^* \in X^*. \tag{2}$$

In what follows, we will investigate the relationships between dual spaces of subspaces and quotient spaces. The overall strategy is to use (1) and (2) repeatedly whenever we can, and think about the Hahn-Banach theorem(s) whenever we meet trouble.

Proposition 1. *{ X is reflexive }* $\iff$ *{ X^* is reflexive }.*

Proof. let $Q\colon X \to X^{**}$ and $\widetilde{Q}\colon X^* \to X^{***}$ be the canonical maps, that is

$$\begin{array}{ccc} X^* & \xrightarrow{\widetilde{Q}} & X^{***} \\ \vdots & & \vdots \\ X & \xrightarrow{Q} & X^{**} \end{array}$$

($\Longrightarrow$) Let $x_0^{***} \in X^{***}$. Define $x_0^* \in X^*$ by

$$\langle x, x_0^* \rangle := \langle Qx, x_0^{***} \rangle \quad \forall\, x \in X. \tag{3}$$

Since X is reflexive, for every $x^{**} \in X^*$ there is a unique $x \in X$ such that

$$x^{**} = Qx, \tag{4}$$

thus

$$\begin{aligned} \langle x^{**}, \widetilde{Q}x_0^* \rangle &= \langle x_0^*, x^{**} \rangle && \text{by (2)} \\ &= \langle x_0^*, Qx \rangle && \text{by (4)} \\ &= \langle x, x_0^* \rangle && \text{by (2)} \\ &= \langle Qx, x_0^{***} \rangle && \text{by (3)} \\ &= \langle x^{**}, x_0^{***} \rangle && \text{by (4).} \end{aligned}$$

Hence $\widetilde{Q}x_0^* = x_0^{***}$ since x^{**} is arbitrary in X^{**}. Since the following argument style is more or less the same, below we avoid indicating the reasons of most steps but only point out the essential steps.

($\Longleftarrow$) Suppose otherwise that $QX \subsetneq X^{**}$. Then by Hahn-Banach theorem (closure point checking), there exists $x_0^{***} \in X^{***}$ such that $x_0^{***} \neq 0$ and

$$\langle Qx, x_0^{***} \rangle = 0 \quad \text{for all } x \in X.$$

On the other hand, there exists $x_0^* \in X^*$ such that $\widetilde{Q}x_0^* = x_0^{***}$ since X^* is reflexive. Then for all $x \in X$

$$\langle x, x_0^* \rangle = \langle x_0^*, Qx \rangle = \langle Qx, \widetilde{Q}x_0^* \rangle = \langle Qx, x_0^{***} \rangle = 0.$$

Hence $x_0^* = 0$ since x is arbitrary. This implies $x_0^{***} = \widetilde{Q}x_0^* = 0$, which contradicts $x_0^{***} \neq 0$. □

Proposition 2. *If X is reflexive, then X/M is reflexive.*

Proof. By Proposition 1 X^* is relexive. By Proposition 3 $(X/M)^* = M^\perp \subset X^*$ is reflexive as a closed subspace of X^*. Hence X/M is reflexive by Proposition 1. □

— THE END OF MAIN CONTENT —

Appendix

Recall the *annihilator*

$$M^\perp := \{x^* \in X^* \colon x^*(m) = 0, \forall\, m \in M\}.$$

Proposition 3. *Let $\pi^* \colon (X/M)^* \to X^*$ be the adjoint operator of the natural projection $\pi \colon X \to X/M$. Then π^* is an isometry, in particular,*

$$M^\perp = \pi^*(X/M)^*.$$

Proof. We focus on the following diagram.

$$\begin{array}{ccc} X & \xrightarrow{\quad \pi \quad} & X/M \\ \vdots & & \vdots \\ X^* \supset M^\perp & \xleftarrow{\ \pi^* \ } & (X/M)^* \end{array}$$

It follows from (1) that

$$\langle x, \pi^* y^* \rangle := \langle \pi x, y^* \rangle \quad \forall\, y^* \in (X/M)^*, \forall\, x \in X.$$

Since $\pi m = 0 \in X/M$ for every $m \in M$, we have for $y^* \in (X/M)^*$ and for $m \in M$,

$$\langle m, \pi^* y^* \rangle = \langle \pi m, y^* \rangle = \langle 0, y^* \rangle = 0.$$

Hence $\pi^*(X/M)^* \subset M^\perp$ since m is arbitrary. On the other hand, for every $x_0^* \in M^\perp$, define $y_0^* \in (X/M)^*$ by

$$\langle \pi x, y_0^* \rangle := \langle x, x_0^* \rangle \quad \forall\, \pi x \in X/M.$$

Then y_0^* is well defined since for every $y \in X$ with $\pi y = \pi x$, that is $x - y \in M$, and so

$$\langle x, x_0^* \rangle - \langle y, x_0^* \rangle = \langle x - y, x_0^* \rangle = 0$$

by $x_0^* \in M^\perp$. Hence

$$\langle x, \pi^* y_0^* \rangle = \langle \pi x, y_0^* \rangle = \langle x, x_0^* \rangle$$

for all $x \in X$, thus $x_0^* = \pi^* y_0^*$. Together we have $\pi^*(X/M)^* = M^\perp$.

The linearity of π^* is obvious. Let $y^* \in (X/M)^*$ such that $\langle x, \pi^* y^* \rangle = 0$ for all $x \in X$. Since π is surjective, for every $y \in X/M$ there exists $x \in X$ with $\pi x = y$, then $\langle y, y^* \rangle = \langle \pi x, y^* \rangle = \langle x, \pi^* y^* \rangle = 0$. Hence $y^* = 0$ since y is arbitrary in X/M. This implies π^* is injective.

Next we check $\|\pi^* y^*\| = \|y^*\|$ for all $y^* \in (X/M)^*$. Since $\|\pi^*\| = \|\pi\| \le 1$ (Hahn-Banach is hidden here), we have $\|\pi^* y^*\| \le \|y^*\|$. Hence it suffices to check $\|y^*\| \le \|\pi^* y^*\|$. For every $m \in M$,

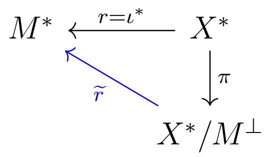
$$|\langle \pi x, y^* \rangle| = |\langle \pi(x+m), y^* \rangle| = |\langle x + m, \pi^* y^* \rangle| \le \|\pi^* y^*\| \|x + m\|.$$

Taking infimum with respect to $m \in M$ gives $|\langle \pi x, y^* \rangle| \le \|\pi^* y^*\| \|\pi x\|$, thus $\|y^*\| \le \|\pi^* y^*\|$. □

Proposition 4. *Let $\widetilde{r} \colon X^*/M^\perp \to M^*$ be the splitting of $\iota^* \colon X^* \to M^*$ along the natural projection $\pi \colon X^* \to X^*/M^\perp$, where ι^* is the adjoint of the natural inclusion $\iota \colon M \to X$. Then $\widetilde{r}$ is an isometry, in particular,*

$$M^* = \widetilde{r}(X^*/M^\perp).$$

Proof. We focus on the following diagram.



Define $\widetilde{r}\colon X^*/M^\perp \to M^*$ by

$$\langle m, r(\pi x^*)\rangle := \langle m, x^*\rangle \quad \forall\, \pi x^* \in X^*/M^\perp, \forall\, m \in M. \tag{5}$$

Then $\widetilde{r}$ is well defined since for every $y^* \in X^*$ with $\pi y^* = \pi x^*$, that is $x^* - y^* \in M^\perp$, and so

$$\langle m, x^*\rangle - \langle m, y^*\rangle = \langle m, x^* - y^*\rangle = 0$$

for all $m \in M$.

The linearity of $\widetilde{r}$ is obvious. Let $\pi x^* \in X^*/M^\perp$ such that $\langle m, \widetilde{r}\pi x^*\rangle = 0$ for all $m \in M$. Then

$$\langle m, x^*\rangle = \langle m, \widetilde{r}\pi x^*\rangle = 0$$

for all $m \in M$. Hence $x^* \in M^\perp$ and so $\pi x^* = 0 \in X^*/M^\perp$. This implies $\widetilde{r}$ is injective.

Let $m^* \in M^*$. By Hahn-Banach theorem, there exists $x^* \in X^*$ such that $m^* = \iota^* x^*$. Then for all $m \in M$,

$$\langle m, \widetilde{r}\pi x^*\rangle = \langle m, x^*\rangle = \langle \iota m, x^*\rangle = \langle m, \iota^* x^*\rangle = \langle m, m^*\rangle.$$

This implies $\widetilde{r}\pi x^* = m^*$ since m is arbitrary. Hence the surjectivity of $\widetilde{r}$ is obtained.

Next we show $\|\widetilde{r}\pi x^*\| = \|\pi x^*\|$ for all $\pi x^* \in X^*/M^\perp$. Since (5) is equivalent to

$$\langle m, r\pi x^*\rangle = \langle \iota m, x^*\rangle = \langle m, \iota^* x^*\rangle$$

for all $m \in M$ and $x^* \in X^*$, we have $\iota^* = \widetilde{r}\pi$. Recall $\|\iota^*\| = \|\iota\| = 1$. Then for $x^* \in X$ and $y^* \in M^\perp$,

$$\|\widetilde{r}\pi x^*\| = \|\widetilde{r}\pi(x^* + y^*)\| = \|\iota^*(x^* + y^*)\| \le \|x^* + y^*\|.$$

Taking infimum with respect to $y^* \in M^\perp$ gives $\|\widetilde{r}\pi x^*\| \le \|\pi x^*\|$. On the other hand, by Hahn-Banach theorem, for each $\widetilde{r}(\pi x^*) \in M^*$ there exists $y^* \in X^*$ such that $\iota^* y^* = \widetilde{r}(\pi x^*) = \iota^* x^*$ and $\|y^*\| = \|\widetilde{r}\pi x^*\|$. Then it follows from $\iota y^* = \iota x^*$ that $x^* - y^* \in M^\perp$, and so $\pi x^* = \pi y^*$. Hence

$$\|\pi x^*\| = \|\pi y^*\| \le \|y^*\| = \|\widetilde{r}\pi x^*\|.$$

□

Proposition 5. *$(M^\perp)^\perp = \iota^{**} M^{**}$.*

Proof. ($\supset$) Let $x_0^{**} \in \iota^{**} M^{**}$, then $x_0^{**} = \iota^{**} m_0^{**}$ for some $m_0^{**} \in M^{**}$. For every $x^* \in M^\perp$ which means $\iota^* x^* = 0 \in M^*$,

$$\langle x^*, x_0^{**}\rangle = \langle x^*, \iota^{**} m_0^{**}\rangle = \langle \iota^* x^*, m_0^{**}\rangle = \langle 0, m_0^{**}\rangle = 0.$$

Hence $x_0^{**} \in (M^\perp)^\perp$ since $x^* \in M^\perp$ is arbitrary.

($\subset$) Let $x_0^{**} \in (M^\perp)^\perp$. By Hahn-Banach theorem for each $m^* \in M^*$ there exists $x^* \in X^*$ with $\iota^* x^* = m^*$. Define $m_0^{**} \in M^{**}$ by

$$\langle m^*, m_0^{**}\rangle := \langle x^*, x_0^{**}\rangle \quad \text{for } m^* \in M^* \text{ with } m^* = \iota^* x^* \text{ for some } x^* \in X^*.$$

Then m_0^{**} is well defined since for $y^* \in X^*$ with $\iota^* y^* = m^* = \iota^* x^*$, that is $x^* - y^* \in M^\perp$, and so

$$\langle x^*, x_0^{**}\rangle - \langle y^*, x_0^{**}\rangle = \langle x^* - y^*, x_0^{**}\rangle = 0$$

by $x_0^* \in (M^\perp)^\perp$. Hence for $x^* \in X^*$,

$$\langle x^*, \iota^{**} m_0^{**}\rangle = \langle \iota^* x^*, m_0^{**}\rangle = \langle x^*, x_0^{**}\rangle.$$

This implies $x_0^{**} = \iota^{**} m_0^{**} \in \iota^{**} M^{**}$. □

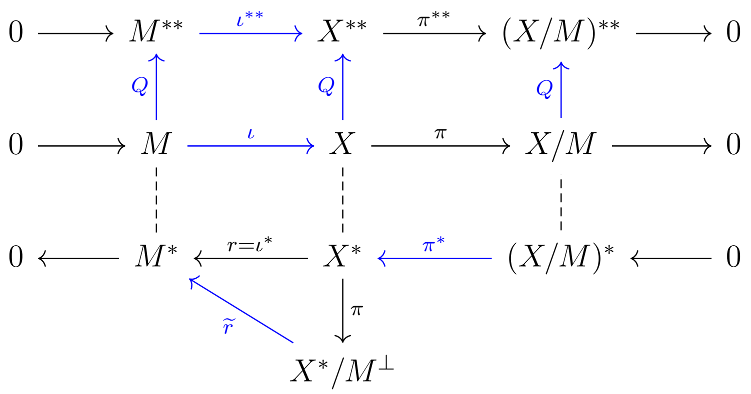
Corollary 6. *$(X/M)^{**} = X^{**}/\iota^{**}M^{**}$.*

Proof. By Proposition 3 and Proposition 4, we have, up to isometric isomorphisms, the following holds

$$(X/M)^{**} = ((X/M)^*)^* = (M^\perp)^* = X^{**}/(M^\perp)^\perp.$$

It follows from Proposition 5 that $(M^\perp)^\perp = \iota^{**}M^{**}$, which completes the proof. In other words, $\pi^{**} \colon X^{**} \to (X/M)^{**} = X^*/\iota^{**}M^{**}$ is the natural projection. □

Theorem 7. *The rows of the following commutative diagram are short exact sequences.*



Proof. It follows from Corollary 6 that $\operatorname{Im} \iota^{**} = \ker \pi^{**}$. Hence the middle node of the top row is exact. On the other hand, the exactness at other nodes is easy to justify. □

Proposition 8. *{ X is reflexive } $\iff$ { M & X/M are reflexive }*

Proof. This follows from Theorem 7 with diagram chasing, especially the short five lemma. □