

## Recall

On a finite dimensional vector space, all the norms are equivalent. All the finite dimensional spaces are isomorphic. For normed spaces, finite dimensionality  $\iff$  locally compactness.

Let  $X, Y$  be normed spaces and  $(T_n)_{n=1}^\infty, T: X \rightarrow Y$  be linear operators.

- $\{ T \text{ continuous} \} \iff \{ T \text{ continuous at } 0 \} \iff \{ T \text{ bounded} \}$ .
- If  $\dim X < \infty$ , then  $T$  must be continuous. Moreover,  $\{ T_n x \rightarrow T x \text{ for all } x \in X \} \iff \{ T_n \xrightarrow{\|\cdot\|} T \}$ . However, the direction  $\implies$  may not hold when  $\dim X = \infty$ .
- If  $\dim Y < \infty$ , then  $\{ T \text{ bounded} \} \iff \{ \ker T \text{ closed} \}$ . In particular, this holds for linear functionals. However, the direction  $\impliedby$  may not hold when  $\dim Y = \infty$ .
- Equivalent definitions of the operator norm

$$\begin{aligned} \|T\| &= \sup\left\{ \frac{\|Tx\|}{\|x\|} : x \in X, \|x\| \neq 0 \right\} \\ &= \sup\{\|Tx\| : x \in X, \|x\| = 1\} \\ &= \sup\{\|Tx\| : x \in X, \|x\| \leq 1\} \\ &= \inf\{M > 0 : \|Tx\| \leq M\|x\|, \forall x \in X\}. \end{aligned}$$

The operator norm depends on both of the norms in the domain  $X$  and in the range  $Y$ .

## Dual space

**Example 1** (Dual-space relationship). Let  $1 \leq p < \infty$  and  $1 < q \leq \infty$  such that  $\frac{1}{p} + \frac{1}{q} = 1$ . Then  $(\ell^p)^* = \ell^q$ .

*Proof.* We begin with some convenient notation. For  $x = (x(i))_{i=1}^\infty \in \ell^p$  and  $y = (y(i))_{i=1}^\infty \in \ell^q$ , define a pairing

$$\langle x, y \rangle := \sum_{i=1}^\infty x(i)y(i). \tag{1}$$

By Hölder's inequality,

$$|\langle x, y \rangle| \leq \sum_{i=1}^\infty |x(i)y(i)| \leq \|x\|_p \|y\|_q < \infty. \tag{2}$$

Hence  $\langle \cdot, \cdot \rangle: \ell^p \times \ell^q \rightarrow \mathbb{K}$ , where  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ . It is readily checked that for  $\alpha \in \mathbb{K}, x, \tilde{x} \in \ell^p$  and  $y \in \ell^q$ ,

$$\langle \alpha x + \tilde{x}, y \rangle = \alpha \langle x, y \rangle + \langle \tilde{x}, y \rangle \text{ and } \langle x, \alpha y + \tilde{y} \rangle = \alpha \langle x, y \rangle + \langle x, \tilde{y} \rangle. \tag{3}$$

By (2) and (3), for any fixed  $y \in \ell^q$ , the map  $\langle \cdot, y \rangle: \ell^p \rightarrow \mathbb{K}$  is continuous and linear, i.e.,  $\langle \cdot, y \rangle \in (\ell^p)^*$ . To prove  $(\ell^p)^* = \ell^q$ , we will show that the map

$$\begin{aligned} T: \ell^q &\rightarrow (\ell^p)^* \\ y &\mapsto \langle \cdot, y \rangle \end{aligned}$$

is an isometric isomorphism.

- (i) (linear and injective) By (3),  $T$  is linear. If  $\langle \cdot, y \rangle$  is identically zero on  $\ell^p$ , then by applying  $\langle \cdot, y \rangle$  to  $e_i$ , where  $e_i(k) = 1$  if  $k = i$  and  $e_i(k) = 0$  if  $k \neq i$ , we get  $y = 0 \in \ell^q$ , thus  $T$  is injective.
- (ii) (surjective) Let  $\Lambda \in (\ell^p)^*$ . We will find  $y \in \ell^q$  such that for all  $x \in \ell^p$ ,  $\Lambda x = \langle x, y \rangle$ . If  $\Lambda = 0$ , then  $y = 0$  satisfying the requirement. Below assume  $\Lambda \neq 0$ . (Recall basis is like the ‘skeleton’ of a vector space. To determine the behavior of a linear map  $\Lambda$  on the whole space, it is often enough to determine the how  $\Lambda$  acts on the basis vectors.) For  $i \in \mathbb{N}$ , let  $e_i$  be the sequence taking 1 on  $i$ -th term and 0 on all the other terms. Define a sequence  $y = (\Lambda e_i)_{i=1}^\infty$ . We will check  $y$  is the desired sequence.

Let  $x \in \ell^p$ . It is readily proved that  $\{e_i\}_{i=1}^\infty$  is a Schauder basis in  $\ell^p$  ( $1 \leq p < \infty$ ). Then  $x = \sum_{i=1}^\infty x(i)e_i \in \ell^p$ . Since  $\Lambda$  is continuous and linear,

$$\Lambda x = \Lambda \left( \sum_{i=1}^\infty x(i)e_i \right) = \sum_{i=1}^\infty x(i)\Lambda e_i = \langle x, y \rangle. \tag{4}$$

Next we check  $y \in \ell^q$ .

When  $q = \infty$ . Suppose on the contrary that  $y \notin \ell^\infty$ . Then there exist  $i_0 \in \mathbb{N}$  such that  $|y(i_0)| > 2\|\Lambda\|$ . However,  $|y(i_0)| = |\Lambda e_{i_0}| \leq \|\Lambda\|$ , which is a contradiction. Hence  $y \in \ell^\infty$ .

When  $q < \infty$ . Define  $x_n = \begin{cases} |y(i)|^{q-1} \exp(-\theta_i) & \text{if } i \leq n \\ 0 & \text{if } i > n \end{cases}$ , where  $y(i) = |y(i)|e^{i\theta_i}$ . Then  $x_n \in \ell^p$ . By (4) and the boundedness of  $\Lambda$ ,

$$\sum_{i=1}^n |y(i)|^q = |\langle x_n, y \rangle| = |\Lambda x_n| \leq \|\Lambda\| \|x_n\|_p = \|\Lambda\| \left( \sum_{i=1}^n |y(i)|^q \right)^{1/p}.$$

Dividing both sides by  $(\sum_{i=1}^n |y(i)|^q)^{1/p}$  (which is nonzero when  $n$  large enough),

$$\left( \sum_{i=1}^n |y(i)|^q \right)^{1/q} \leq \|\Lambda\|.$$

Letting  $n \rightarrow \infty$  gives  $\|y\|_q \leq \|\Lambda\|$ , thus  $y \in \ell^q$ .

- (iii) (isometric) Let  $y \in \ell^q$ . Note that (2) implies  $\|Ty\| \leq \|y\|_q$ . If  $y = 0$ , then  $Ty = 0$ . Below assume  $y \neq 0$ .

When  $q = \infty$ . For any  $\varepsilon > 0$ , there exists  $i \in \mathbb{N}$  such that  $|y(i)| \geq \|y\|_\infty - \varepsilon$ . Hence

$$|\langle e_i, y \rangle| = |y(i)| \geq \|y\|_\infty - \varepsilon.$$

Letting  $\varepsilon \rightarrow 0$  and since  $\|e_i\|_1 = 1$  for all  $i \in \mathbb{N}$ , we have  $\|Ty\| \geq \|y\|_\infty$ .

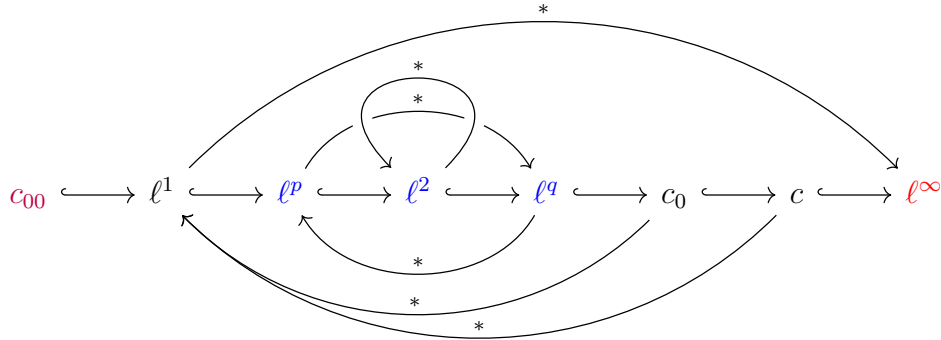
When  $q < \infty$ . Write  $y(i) = |y(i)| \exp(i\theta_i)$  for  $i \in \mathbb{N}$ . Define the ‘conjugate function’

$$y^* = \|y\|_q^{1-q} \left( |y(i)|^{q-1} \exp(-i\theta_i) \right)_{i=1}^\infty. \tag{5}$$

Then  $\|y^*\|_p = 1$  and  $\langle y^*, y \rangle = \|y\|_q$ . Hence  $\|Ty\| \geq \|y\|_q$ .

□

We can summarize the dual-space relationships of sequence spaces. Recall  $c_0$  is the space of sequences converging to zero and  $c$  is the space of convergent sequences, both of which are equipped with sup-norm while considering as normed spaces. For vector spaces  $A$  and  $B$ , denote  $A \hookrightarrow B$  if  $A \subset B$ . For Banach spaces  $X$  and  $Y$ , denote  $X \xrightarrow{*} Y$  if  $Y = X^*$ . Recall  $(c_0)^* = c^* = \ell^1$ . Let  $1 < p < 2$  and  $2 < q < \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$ . Then



*Remark.* So far,  $\ell^\infty$  is the only sequence space whose dual space is not yet characterized. One reason why our justification in [Example 1](#) doesn't work is that  $\ell^\infty$  is not separable, thus lack of a Schauder basis. However, in the future lectures we may see that we can decompose  $(\ell^\infty)^*$  into a direct sum of  $\ell^1$  and another space.

Below is a simple application of the representation of  $(\ell^2)^*$ .

**Example 2.** For  $x = (x(i))_{i=1}^\infty \in \ell^2$ , define  $\Lambda x = \sum_{i=1}^\infty \frac{x(2i)}{i}$ . Show that  $\Lambda \in (\ell^2)^*$  and compute  $\|\Lambda\|$ .

*Proof.* For  $i \in \mathbb{N}$ , define

$$y(i) = \Lambda(e_i) = \begin{cases} \frac{1}{k} & , i = 2k, \\ 0 & , i = 2k - 1, \end{cases}$$

where  $\{e_i\}_{i=1}^\infty$  is the standard Schauder basis of  $\ell^2$ . Let  $y = (y(i))_{i=1}^\infty$ . Then  $\Lambda(\cdot) = \langle \cdot, y \rangle$ .

Since  $\|y\|_2 = (\sum_{k=1}^\infty \frac{1}{k^2})^{1/2} = \pi/\sqrt{6}$ , it follows from [Example 1](#) that  $\Lambda \in (\ell^2)^*$  and  $\|\Lambda\| = \|y\|_2 = \pi/\sqrt{6}$ . □