General information

- Tutor: Zhou Feng (zfeng@math.cuhk.edu.hk);
- Time and venue: Th. 11:30–12:15, ERB 804;
- Course webpage: `https://www.math.cuhk.edu.hk/course/2223/math4010`
- References:
  - **Lecture Notes** of Prof. Leung (available on course webpage).
  - **Textbook:** S. Ovchinnikov, Functional analysis, Springer, (2018).
  - E. Kreyszig, Introductory functional analysis with applications, John Wiley & Thusns (1978).
  - W. Rudin, Functional analysis, McGraw-Hill, (1991).

Recall

A normed space is a vector space equipped with a compatible norm (1. **non-degenerate** positivity 2. scaling property 3. triangle inequality). A Banach space is a **complete** normed space. Besides the definition by the convergence of Cauchy sequence, completeness can be characterized via series.

Every normed space has a unique completion to a Banach space. Every Banach space with Schauder basis is separable but the converse is false (P. Enflo 1973) (countering to the case in Hilbert spaces).

Normed & Banach spaces

Example 1. Show that the space of *bounded variation functions* $BV[a,b]$ is a normed space with the norm

$$\|x\| = |x(a)| + V(x) \qquad \text{for } x \in BV[a,b],$$

where $V(x)$ denotes the *total variation* of x on $[a,b]$.

We first introduce the bounded variation functions. Let $[a,b]$ be a bounded closed interval in $\mathbb{R}$. Let $x\colon [a,b] \to \mathbb{C}$ be a complex-valued function. For any partition $P = \{a = t_0 \le \cdots \le t_n = b\}$, define the *variation* of x with respect to P by

$$V(x,P) := \sum_{k=1}^{n} |x(t_k) - x(t_{k-1})|.$$

Then the *total variation* of x is defined as

$$V(x) := \sup\{V(x,P)\colon P \text{ is a partition of } [a,b]\}.$$

Finally the set of *bounded variation functions* is

$$BV[a,b] := \{x\colon [a,b] \to \mathbb{C} \,|\, V(x) < \infty\}.$$

Proof. First we show $BV[a, b]$ that is closed under addition and scalar product, thus making it a vector space. Let $P = \{a = t_0 \le \cdots \le t_n = b\}$ be any partition of $[a, b]$. For any $x, y \in BV[a, b]$ and $\alpha \in \mathbb{C}$, the scaling property of $|\cdot|$ implies that

$$V(\alpha x, P) = \sum_{k=1}^{n} |\alpha x(t_k) - \alpha x(t_{k-1})| = |\alpha| \sum_{k=1}^{n} |x(t_k) - x(t_{k-1})| = |\alpha| V(x, P)$$

and the triangle inequality of $|\cdot|$ implies that

$$\begin{aligned} V(x + y, P) &= \sum_{k=1}^{n} |x(t_k) + y(t_k) - x(t_{k-1}) - y(t_{k-1})| \\ &\le \sum_{k=1}^{n} |x(t_k) - x(t_{k-1})| + |y(t_k) - y(t_{k-1})| \\ &= \sum_{k=1}^{n} |x(t_k) - x(t_{k-1})| + \sum_{k=1}^{n} |y(t_k) - y(t_{k-1})| = V(x, P) + V(y, P). \end{aligned}$$

Taking supremum with respect to partition P gives

$$V(\alpha x) = \sup_P V(\alpha x, P) = \sup_P \big(|\alpha| V(x, P)\big) = |\alpha| \sup_P V(x, P) = |\alpha| V(x) \tag{1}$$

and

$$\begin{aligned} V(x + y) &= \sup_P V(x + y, P) \\ &\le \sup_P \big(V(x, P) + V(y, P)\big) \\ &\le \sup_P V(x, P) + \sup_P V(y, P) = V(x) + V(y). \end{aligned} \tag{2}$$

Hence $BV[a, b]$ is closed under addition and scalar product by (1) and (2), thus a vector space.

Next we check that $\|\cdot\|$ is indeed a norm.

- If $\|x\| = |x(a)| + V(x) = 0$, then $x(a) = 0$ and $V(x) = 0$. For any $t \in [a, b]$, consider the partition $P = \{a \le t \le b\}$, then

  $$|x(t) - x(a)| + |x(b) - x(t)| = V(x, P) \le V(x) = 0,$$

  which forces $|x(t) - x(a)| = 0$, that is $x(t) = x(a) = 0$. Hence $x = 0$ since t is arbitrary.
- Let $x \in BV[a, b]$ and $\alpha \in \mathbb{C}$. By the scaling property of $|\cdot|$ and (1),

  $$\|\alpha x\| = |\alpha x(a)| + V(\alpha x) = |\alpha||x(t)| + |\alpha| V(x) = |\alpha|\big(|x(t)| + V(x)\big) = |\alpha| V(x).$$

- Let $x, y \in BV[a, b]$. By the triangle inequality of $|\cdot|$ and (2),

  $$V(x + y) = |x(a) + y(a)| + V(x + y) \le |x(a)| + |y(a)| + V(x) + V(y) = \|x\| + \|y\|.$$

Together we conclude that $(BV[a, b], \|\cdot\|)$ is a normed space. □