THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2022-23 Term 1

Solution to the Test on 21 Nov 2022

1. (*20 points*) We call a normed space X strictly convex if and only if for any $x, y \in S_X$ (the unit sphere of X) with $x \neq y$, we have $\|\frac{1}{2}x + \frac{1}{2}y\| < 1$.

   (a) Show that a normed space X is strictly convex if and only if for $u \in S_X$ and $v \in X$ with $\|u + v\| \leq 1$ and $\|u - v\| \leq 1$ implies $v = 0$.

   (b) Determine whether the following spaces are strictly convex or not.

   (i) $(\mathbb{R}^2, \|\cdot\|_2)$ and (ii) $(C_{\mathbb{R}}[0,1], \|\cdot\|_\infty)$.

Proof.

(a) Since $\|\frac{1}{2}x + \frac{1}{2}y\| \leq \frac{1}{2}\|x\| + \frac{1}{2}\|y\| = 1$ for $x, y \in S_X$, it follows from the definition that X is strictly convex if and only if for $x, y \in S_X$, we have $\|\frac{1}{2}x + \frac{1}{2}y\| = 1$ implies $x = y$.

($\Longrightarrow$) Define $x := u+v$ and $y := u-v$. Then $\|x\| \leq 1$ and $\|y\| \leq 1$. By $\frac{1}{2}x+\frac{1}{2}y = u \in S_X$,

$$1 = \|u\| = \|\frac{1}{2}x + \frac{1}{2}y\| \leq \frac{1}{2}\|x\| + \frac{1}{2}\|y\| \leq 1.$$

This shows $\|x\| = \|y\| = 1$. Since $\|\frac{1}{2}x + \frac{1}{2}y\| = \|u\| = 1$ and X is strictly convex, we have $x = y$. Hence $v = (x-y)/2 = 0$.

($\Longleftarrow$) Let $x, y \in S_X$ such that $\|\frac{1}{2}x + \frac{1}{2}y\| = 1$. Define $u := \frac{x+y}{2}$ and $v := \frac{x-y}{2}$. Then $u \in S_X$. Moreover,

$$\|u+v\| = \|x\| = 1 \quad \text{and} \quad \|u-v\| = \|y\| = 1.$$

By the assumption, we have $v = 0$. This means $x = y$. Hence X is strictly convex.

(b) (i) Yes. Let $x, y \in \mathbb{R}^2$ with $\|x\|_2 = \|y\|_2 = 1$ and $x \neq y$. Then $\left\|\frac{x-y}{2}\right\|_2^2 > 0$. By Parallelogram Law,

$$\left\|\frac{x+y}{2}\right\|_2^2 = \frac{1}{2}\|x\|_2^2 + \frac{1}{2}\|y\|_2^2 - \left\|\frac{x-y}{2}\right\|_2^2 \leq 1 - \left\|\frac{x-y}{2}\right\|_2^2 < 1.$$

This shows that $(\mathbb{R}^2, \|\cdot\|_2)$ is strictly convex.

(ii) No. Consider $f = 1$ on $[0,1]$ and

$$g(x) = \begin{cases} 2x & \text{if } x \in [0, 1/2] \\ 1 & \text{if } x \in (1/2, 1]. \end{cases}$$

Then $f, g \in C_{\mathbb{R}}[0,1]$ with $\|f\|_\infty = \|g\|_\infty = 1$ and $f \neq g$. However, $\left\|\frac{1}{2}f + \frac{1}{2}g\right\|_\infty = 1$. Hence $(C_{\mathbb{R}}[0,1], \|\cdot\|_\infty)$ is not strictly convex.

□

2. (*20 points*) Let X be a Banach space.

   (a) Show that X is reflexive if and only its dual space X^* is reflexive too.

   (b) Does the above statement hold if X is not a Banach space?

Proof.

(a) Let $Q\colon X \to X^{**}$ and $\widetilde{Q}\colon X^* \to X^{***}$ be the canonical maps. We shall argue using the dual pair notation $\langle \cdot, \cdot \rangle$.

$(\implies)$ Let $x_0^{***} \in X^{***}$. Define $x_0^* \in X^*$ by

$$\langle x, x_0^* \rangle := \langle Qx, x_0^{***} \rangle \quad \text{for } x \in X. \tag{1}$$

Since X is reflexive, for $x^{**} \in X^{**}$ there is $x \in X$ such that $x^{**} = Qx$. Then

$$\langle x^{**}, \widetilde{Q}x_0^* \rangle = \langle x_0^*, x^{**} \rangle = \langle x_0^*, Qx \rangle = \langle x, x_0^* \rangle = \langle Qx, x_0^{***} \rangle = \langle x^{**}, x_0^{***} \rangle,$$

where the second last equality is by (1). This implies $\widetilde{Q}x_0^* = x_0^{***}$. Hence X^* is reflexive.

$(\impliedby)$ Suppose on the contrary that $QX \subsetneq X^{**}$. By Hahn-Banach theorem, there exists $x_0^{***} \in X^{***}$ such that $x_0^{***} \neq 0$ and

$$\langle Qx, x_0^{***} \rangle = 0 \quad \text{for } x \in X.$$

Since X^* is reflexive, there exists $x_0^* \in X^*$ such that $\widetilde{Q}x_0^* = x_0^{***}$. Then for $x \in X$,

$$\langle x, x_0^* \rangle = \langle x_0^*, Qx \rangle = \langle Qx, \widetilde{Q}x_0^* \rangle = \langle Qx, x_0^{***} \rangle = 0.$$

This implies $x_0^* = 0$. Hence $x_0^{***} = \widetilde{Q}x_0^* = 0$, which contradicts $x_0^{***} \neq 0$.

(b) No, since a reflexive space is necessarily complete. Let Y be any reflexive space and X be a dense proper subspace of Y. It follows from (a) that Y^* is reflexive. Then $X^* = Y^*$ is reflexive since X is dense in Y. However, X is not reflexive since X is not complete. For example, $Y = (\ell^2, \|\cdot\|_2)$ and $X = (c_{00}, \|\cdot\|_2)$.

□

3. (*20 points*) Let X be a Hilbert space.

(a) Let T be a selfadjoint bounded linear operator. Show that if $T \neq 0$, then $T^n \neq 0$ for all positive integers n. Give an example of a non-selfadjoint operator so that the statement does not hold.

(b) Let $S, T\colon X \to X$ be the linear operators such that $(Tx, y) = (x, Sy)$ for all $x, y \in X$. Show that S and T are both bounded.

Proof.

(a) Let $n \in \mathbb{N}$. Suppose $T^{n+1} = 0$. Then for $x \in X$,

$$T^{2n}x = T^{n-1}(T^{n+1}x) = T^{n-1}(0) = 0.$$

Let y be in the range of T^n. Then $y = T^n x$ for some $x \in X$. Since T is self-adjoint,

$$(y, y) = (T^n x, T^n x) = (x, T^{2n}x) = (x, 0) = 0.$$

Hence $y = 0$, and so $T^n = 0$. This shows that if $T^{n+1} = 0$, then $T^n = 0$. By induction, if $T^n = 0$ for some $n \in \mathbb{N}$, then $T = 0$.

Consider $S = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ on the Hilbert space $\mathbb{C}^2$. Then $S \neq S^*$ and $S^2 = 0$.

(b) Let $x_n \xrightarrow{\|\cdot\|} x$ and $Tx_n \xrightarrow{\|\cdot\|} y$ in X. Then for $z \in H$, by the continuity of the inner product,

$$(Tx, z) = (x, Sz) = \lim_{n\to\infty} (x_n, Sz) = \lim_{n\to\infty} (Tx_n, z) = (y, z),$$

where we have used the assumption in the first and the third inequalities. This shows $Tx = y$. Hence T is bounded by Closed Graph Theorem.

Similarly, let $x_n \xrightarrow{\|\cdot\|} x$ and $Sx_n \xrightarrow{\|\cdot\|} y$ in X. Then for $z \in H$,

$$(z, Sx) = (Tz, x) = \lim_{n\to\infty} (Tz, x_n) = \lim_{n\to\infty} (z, Sx_n) = (z, y).$$

This shows $Sx = y$. Hence S is bounded by Closed Graph Theorem.

□

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