THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2022-23 Term 1

Solution to Course Examination

1. (*10 points*) Let (x_n) be a sequence in the real null sequence space c_0. Show that the sequence (x_n) is weakly convergent in c_0, then $(\|x_n\|_\infty)$ is bounded and $\lim_{n\to\infty} x_n(k)$ is convergent in $\mathbb{R}$ for all $k = 1, 2, \ldots$

   Proof. Let $Q\colon c_0 \to c_0^{**}$ denote the canonical map. Since (x_n) is weakly convergent, then $(Qx_n(x^*)) = (x^*(x_n))$ is convergent for $x^* \in c_0^*$, thus bounded for $x^* \in c_0^*$. Since c_0^* is complete, Uniform Boundedness Theorem implies some $M > 0$ such that

   $$\sup_n \|x_n\|_\infty = \sup_n \|Qx_n\|_\infty \leq M.$$

   This shows that $(\|x_n\|)$ is bounded.

   For $k \in \mathbb{N}$, define

   $$f_k(x) = x(k) \quad \text{for } x = (x(i)) \in c_0.$$

   It is readily checked that $f_k \in c_0^*$. By the weakly convergence of (x_n),

   $$\lim_{n\to\infty} x_n(k) = \lim_{n\to\infty} f_k(x_n)$$

   exists in $\mathbb{R}$. This means that $(x_n(k))$ is convergent for $k \in \mathbb{N}$. □

2. (*10 points*) Let X be a normed space. A subset A of X is said to weakly closed if it satisfies the condition: an element $x_0 \in A$ whenever if for any $\varepsilon > 0$ and any $f_1, \dots, f_n \in X^*$, then there is an element $a \in A$ such that $|f_k(x_0) - f_k(a)| < \varepsilon$ for $k = 1, \dots, n$.

   (i) Show that if A is a weakly closed subset of X, then it is normed closed. Find an example so that the converse does not hold.

   (ii) If we further assume that A is a vector subspace, show that the converse of Part (i) hold.

Proof.

(i) It suffices to prove that the complement A^c is normed open. Let $y \in A^c$. Since A is weakly closed and y is not in A, there exists $\varepsilon > 0$ and $f_1, \dots, f_n \in X^*$ such that

$$y \in \bigcap_{k=1}^{n} \{x \in X \colon |f_k(x_0) - f_k(x)| < \varepsilon\} \subset A^c.$$

Since $f_1, \dots, f_n$ are continuous, it follows that $\bigcap_{k=1}^{n} \{x \in X \colon |f_k(x_0) - f_k(x)| < \varepsilon\}$ is normed open. This shows that y is an interior point of A^c. Hence A^c is normed open, thus A is normed closed.

Consider $X = (c_0, \|\cdot\|_\infty)$ and

$$B := \{x \in c_0 \colon \|x\|_\infty = 1\}.$$

Then B is normed closed since $\|\cdot\|_\infty \colon c_0 \to \mathbb{R}$ is normed continuous. Next we show that B is not weakly closed. Suppose on the contrary that B is weakly closed. Since $c_0^* = \ell^1$, we have $\lim_{n\to\infty} f(e_m) = 0$ for $f \in c_0^*$, where $e_m(i) = \begin{cases} 1 & \text{if } i = m \\ 0 & \text{if } i \neq m \end{cases}$. Then for every $\varepsilon > 0$ and every $f_1, \dots, f_n \in c_0^*$, there exists $e_m \in B$ with m large enough such that

$$|f_k(0) - f_k(e_m)| = |f_k(e_m)| < \varepsilon \quad \text{for all } k = 1, \dots, n.$$

Then $0 \in B$ by the definition of weakly closedness, which contradicts $0 \notin B$.

(ii) Let A be a normed closed subspace of X. If $A = X$, then A is trivially weakly closed. Next we suppose $A \subsetneq X$. Let $y \in X \setminus A$. Since A is a convex normed closed set, Hyperplane Separation Theorem (or Hahn-Banach Theorem) implies some $f \in X^*$ such that

$$\sup\{f(x) \colon x \in A\} < f(y).$$

Taking $\varepsilon < |f(y) - \sup\{f(x) \colon x \in A\}|$ gives

$$\{x \in X \colon |f(y) - f(x)| < \varepsilon\} \bigcap A = \emptyset.$$

Hence there is no $a \in A$ such that $|f(y) - f(a)| < \varepsilon$. This shows that A is weakly closed since we have checked the contrapositive statement of the definition of weakly closedness.

□

3. (*10 points*) Let X be a Hilbert space and let (x_n) be an orthogonal sequence in X. Show that the series $\sum x_n$ is convergent in X if and only if $\sum\langle x_n, y\rangle$ is convergent for all $y \in X$ if and only if $\sum \|x_n\|^2 < \infty$.

Proof.

- Suppose $\sum x_k$ is convergent. Then for $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq m \geq N$,
$$\left\|\sum_{k=m}^{n} x_k\right\| \leq \varepsilon.$$
Let $y \in X$. By Cauchy-Schwarz inequality,
$$\left|\sum_{k=m}^{n}\langle x_k, y\rangle\right| = \left|\left\langle \sum_{k=m}^{n} x_k, y\right\rangle\right| \leq \left\|\sum_{k=m}^{n} x_k\right\| \|y\| \leq \varepsilon\|y\|.$$
Hence $\sum\langle x_k, y\rangle$ is convergent for $y \in X$ since the scalar field is complete.
- Suppose $\sum\langle x_k, y\rangle$ is convergent for $y \in X$. Then $\sum\langle y, x_k\rangle$ is convergent for $y \in X$ since $\langle y, x\rangle = \overline{\langle x, y\rangle}$ for $x, y \in X$.
For $n \in N$, define
$$f_n(y) := \sum_{k=1}^{n}\langle y, x_k\rangle = \langle y, \sum_{k=1}^{n} x_k\rangle \quad \text{for } y \in X.$$
By Cauchy-Schwarz inequality, $f_n \in X^*$ and $\|f_n\| = \|\sum_{k=1}^{n} x_k\|$. Since $\sum\langle y, x_k\rangle$ is convergent for $y \in X$, we have $(f_n(y))$ is bounded for $y \in X$. Then Uniform Boundedness Theorem implies some $M > 0$ such that
$$\sup_n \|f_n\| \leq M.$$
Thus $\sup_n \|f_n\|^2 \leq M^2$. By the orthogonality of (x_k) and Pythagoras Theorem,
$$\sum_{k=1}^{\infty} \|x_k\|^2 = \sup_n \sum_{k=1}^{n} \|x_k\|^2 = \sup_n \|f_n\|^2 \leq M^2.$$
Hence $\sum \|x_k\|^2 < \infty$.
- Suppose $\sum \|x_k\|^2 < \infty$. Then for $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq m \geq N$,
$$\sum_{k=m}^{n} \|x_k\|^2 \leq \varepsilon.$$
By the orthogonality of (x_k) and Pythagoras Theorem,
$$\left\|\sum_{k=m}^{n} x_k\right\|^2 = \sum_{k=m}^{n} \|x_k\|^2 \leq \varepsilon.$$
Hence $\sum x_k$ is convergent since X is complete.

□

4. (*20 points*) Let X be a Hilbert space and let $L(X)$ be the space of all bounded linear operators. We call an element $S \in L(X)$ *bounded below* if $\inf\{\|S(x)\|: x \in X;\ \|x\| = 1\} > 0$.

   (i) Let $S \in L(X)$. Show that S is invertible in $L(X)$ if and only if S is bounded below and the image $S(X)$ is dense in X.

   (ii) Let $T \in L(X)$. Show that the spectrum $\sigma(T)$ of T is exactly the union

$$\{\mu \in \mathbb{C}: T - \mu \text{ is not bounded below}\} \bigcup \{\mu \in \mathbb{C}: \overline{\mu} \text{ is an eigenvalue of } T^*\}.$$

Proof.

(i) ($\Longrightarrow$) Since S is invertible in $L(X)$, there exists $T \in L(X)$ such that $ST = TS = I$ where I denotes the identity map. Hence $S(X) = X$ because $X = ST(X) \subset S(X) \subset X$. For $x \in X$ with $\|x\| = 1$,

$$1 = \|x\| = \|TS(x)\| \leq \|T\|\|S(x)\|.$$

Then

$$\inf\{\|S(x)\|: x \in X;\ \|x\| = 1\} \geq \|T\|^{-1} > 0.$$

This shows that S is bounded below.

($\Longleftarrow$) Since S is bounded below, by definition there exists $\delta > 0$ such that

$$\delta\|x\| \leq \|S(x)\| \quad \text{for all } x \in X. \tag{1}$$

Then S is injective because $\|x\| \leq (1/\delta)\|S(x)\| = 0$ if $S(x) = 0$.

Next we show that $S(X)$ is closed. Let $(S(x_n))$ be a Cauchy sequence in $S(X)$. From (1) we see that (x_n) is also a Cauchy sequence. Since X is complete, there exists $x \in X$ such that $x_n \xrightarrow{\|\cdot\|} x$ in X. By the continuity of S,

$$\lim_{n\to\infty} S(x_n) = S(\lim_{n\to\infty} x_n) = S(x) \in S(X).$$

This shows that $S(X)$ is complete, and so closed in X. Then $S(X) = X$ since $S(X)$ is assumed to be dense in X. Hence Open Mapping Theorem (or Bounded Inverse Theorem) implies that S is invertible in $L(X)$.

(ii) By (i),

$$\begin{aligned}\sigma(T) &= \{\mu \in \mathbb{C}: T - \mu \text{ is not invertible}\} \\ &= \{\mu \in \mathbb{C}: T - \mu \text{ is not bounded below}\} \bigcup \{\mu \in \mathbb{C}: (T - \mu)(X) \text{ is not dense}\},\end{aligned}$$

On the other hand, note that

$$\overline{(T - \mu)(X)} = \left(((T - \mu)(X))^\perp\right)^\perp = (\ker(T - \mu)^*)^\perp = (\ker(T^* - \overline{\mu}))^\perp.$$

Then

$$\begin{aligned}\{\mu \in \mathbb{C}: (T - \mu)(X) \text{ is not dense}\} &= \{\mu \in \mathbb{C}: \overline{(T - \mu)(X)} \neq X\} \\ &= \{\mu \in \mathbb{C}: (\ker(T^* - \overline{\mu}))^\perp \neq X\} \\ &= \{\mu \in \mathbb{C}: \ker(T^* - \overline{\mu}) \neq 0\} \\ &= \{\mu \in \mathbb{C}: \overline{\mu} \text{ is an eigenvalue of } T^*\}.\end{aligned}$$

Together, we have finished the proof.

□

— *THE END* —