THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2022-23 Term 1
Solution to Homework 5

1. We say that that two non-empty subsets A and B of a vector space X may be *separated by a hyperplane* if there is a linear functional $f\colon X \to \mathbb{R}$ and $c \in \mathbb{R}$ such that

$$f(x) < c \text{ for } x \in A \quad \text{and} \quad f(x) > c \text{ for } x \in B.$$

Let c_{00} denote the space of finite real sequences, that is

$$c_{00} = \{(x(i)) \in \mathbb{R}^{\mathbb{N}} \colon \text{there exists } n \in \mathbb{N} \text{ such that } x(i) = 0 \text{ for all } i > n\}.$$

Let $M \subset c_{00}$ be the set of sequences whose leading nonzero term is positive, that is

$$M := \{(x(i)) \in c_{00} \colon \text{there exists } n \in \mathbb{N} \text{ such that } x(n) > 0 \text{ and } x(i) = 0 \text{ for all } i < n\}.$$

Show that the sets M and $-M$ are convex and disjoint, but they cannot be separated by a hyperplane. (Recall $c_{00}^* = \ell_1$.)

Proof. It is readily checked from the definition that M and $-M$ are convex sets and disjoint to each other. Suppose otherwise that there exists a linear functional f on c_{00} and $c \in \mathbb{R}$ such that

$$f > c \text{ on } M \quad \text{and} \quad f < c \text{ on } -M. \tag{1}$$

Let $\{e_n\}_{n\in\mathbb{N}}$ denote the standard base of c_{00}, that is $e_n(n) = 1$ and $e_n(i) = 0$ for $i \neq n$. For $k \in \mathbb{Z}$, define $x_k \in M$ by

$$x_k(i) = \begin{cases} 1 & \text{if } i = 1 \\ k & \text{if } i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

Then by (1),

$$f(x_k) = f(e_1) + kf(e_2) > c$$

for all $k \in \mathbb{Z}$. This forces $f(e_2) = f(-e_2) = 0$. However, it follows from the definition that $e_2 \in M$ and $-e_2 \in -M$, which contradicts (1). □

2. Let X and Y be Banach spaces and $T\colon X \to Y$ a one-to-one bounded linear operator. Show that $T^{-1}\colon T(X) \to X$ is bounded if and only if $T(X)$ is closed in Y.

Proof. ($\Longrightarrow$) Let (y_n) be a Cauchy sequence in $T(X)$. Since T^{-1} is bounded, then $\|T^{-1}y\| \leq \|T^{-1}\|\|y\|$ for all $y \in T(X)$. This implies that $(T^{-1}y_n)$ is a Cauchy sequence in X. By the completeness of X, we have $\lim_{n\to\infty} T^{-1}y_n = x$ for some $x \in X$. Then since T is continuous,

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} T(T^{-1}y_n) = T(\lim_{n\to\infty} T^{-1}y_n) = Tx.$$

This shows that $T(X)$ is complete, thus closed in Y.

($\Longleftarrow$) Since $T(X)$ is closed and Y is Banach, then $T(X)$ is a Banach space. Then T is an open mapping by Open Mapping Theorem. It follows that the preimage of every open set under T^{-1} is open. Hence T^{-1} is continuous, that is, bounded. □

— *THE END* —