THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4010 Functional Analysis 2022-23 Term 1

Solution to Homework 1

1. Let $B(S)$ be the vector space of all bounded $\mathbf{F}$-valued functions on a nonempty set S. Define

$$\|x\| = \sup\{|x(t)| : t \in S\}.$$

Prove that $(B(S), \|\cdot\|)$ is a complete normed space (cf. [Textbook, Theorem 3.5]).

Proof. First we verify the axioms of a norm. Let $x, y \in B(S)$. Then $\|x\|, \|y\| < \infty$ by the definition of $B(S)$.

- If $\|x\| = 0$, then for all $t \in S$, $|x(t)| \leq \|x\| = 0$ which implies $x(t) = 0$. Hence $x = 0$ on S.
- Let $\alpha \in \mathbf{F}$. Then $\|\alpha x\| = \sup\{|\alpha x(t)| : t \in S\} = |\alpha| \sup\{|x(t)| : t \in S\} = |\alpha|\|x\|$.
- $\|x+y\| = \sup_{t\in S} |(x+y)(t)| \leq \sup_{t\in S}(|x(t)|+|y(t)|) \leq \sup_{t\in S} |x(t)|+\sup_{t\in S} |y(t)| = \|x\|+\|y\|$, where the first inequality is by the triangle inequality of $|\cdot|$ in $\mathbf{F}$.

Next we establish the completeness. Let (x_n) be a Cauchy sequence in $B(S)$. Then for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$,

$$|x_m(t) - x_n(t)| \leq \|x_m - x_n\| \leq \varepsilon \tag{1}$$

for all $t \in S$. Hence for each $t \in S$, $(x_n(t))$ is a Cauchy sequence in $\mathbf{F}$. By the completeness of $\mathbf{F}$, for each $t \in S$ there exists $x(t) \in \mathbf{F}$ such that $\lim_{n\to\infty} x_n(t) = x(t)$. Define x by setting value $x(t)$ for each $t \in S$. Then let $m \to \infty$ in (1), we have

$$|x(t) - x_n(t)| \leq \varepsilon$$

for all $t \in S$ when n is large enough, that is $\lim_{n\to\infty} \|x - x_n\| = 0$. Moreover, since $\|x\| \leq \|x_n\|+\|x-x_n\| \leq \|x_n\|+1 < \infty$ when n large, we have $x \in B(S)$. Hence $B(S)$ is complete. □

2. Show that for $1 \leq p < \infty$,

$$c_{00} \subset \ell_p \subset c_0 \subset c \subset \ell_\infty$$

and all inclusions are proper. (Please refer to [Textbook, Section 3.3 & 3.4] for the definitions of above spaces.)

Also show that

(a) c_{00} is dense in the space c_0,

(b) c_{00} is dense in ℓ_p,

(c) c_{00} is not dense in c,

(d) c_{00} is not dense in ℓ_∞,

in the topology defined by the sup-norm, $\|\cdot\|_\infty$.

Proof. First we check the strict inclusions.

- Let $x = (x(i))_{i=1}^{\infty} \in c_{00}$. Then there exists $n \in \mathbb{N}$ such that $x(i) = 0$ for all $i \geq n+1$. Thus
$$\|x\|_p^p = \sum_{i=1}^{\infty} |x(i)|^p = \sum_{i=1}^{n} |x(i)|^p < \infty,$$
which implies $x \in \ell_p$. Hence $c_{00} \subset \ell_p$. On the other hand, take $y = (1/i^{2p})_{i=1}^{\infty}$. Then $\|y\|_p^p = \sum_{i=1}^{\infty} 1/i^2 < \infty$, which implies $y \in \ell_p$, but $y \notin c_{00}$.
- Let $x = (x(i))_{i=1}^{\infty} \in \ell_p$. Then $\|x\|_p^p = \sum_{i=1}^{\infty} |x(i)|^p < \infty$, which implies $\lim_{i\to\infty} |x(i)| = 0$. This means $x \in c_0$. Hence $\ell_p \subset c_0$. On the other hand, take $y = (1/i^p)_{i=1}^{\infty}$. Then $\|y\|_p^p = \sum_{i=1}^{\infty} 1/i = \infty$, which implies $y \notin \ell_p$, but $y \in c_0$.
- Let $x = (x(i))_{i=1}^{\infty} \in c_0$. Then $\lim_{i\to\infty} |x(i)| = 0$, thus $(x(i))$ converges to 0. Hence $x \in c$ and $c_0 \subset c$. However, the constant sequence $y = (1)_{i=1}^{\infty} \in c$ but $y \notin c_0$.
- Since every convergent sequence is bounded, we have $c \subset \ell_\infty$. However, not every bounded sequence is convergent, for example, $y = ((-1)^i)_{i=1}^{\infty} \in \ell_\infty$ but $y \notin c$.

Notice that (a) implies (b) by $\ell_p \subset c_0$ and (c) implies (d) by $c \subset \ell_\infty$. It suffices to check (a) and (c).

(a) Let $x = (x(i))_{i=1}^{\infty} \in c_0$. Then $\lim_{i\to\infty} |x(i)| = 0$. For every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $|x(i)| \leq \varepsilon$ for all $i \geq n$. Define $y = (y(i))_{i=1}^{\infty}$ with $y(i) = x(i)$ for $i < n$ and $y(i) = 0$ for $i \geq n$. Then
$$\|x - y\|_\infty = \sup_{i\in\mathbb{N}} |x(i) - y(i)| = \sup_{i\geq n} |x(i)| \leq \varepsilon.$$
Hence c_{00} is dense in c_0 with respect to $\|\cdot\|_\infty$.

(c) Take the constant sequence $x = (1)_{i=1}^{\infty} \in c$. Then for every $y = (y(i))_{i=1}^{\infty} \in c_{00}$, there exists $n \in \mathbb{N}$ such that $y(i) = 0$ for all $i \geq n$. Hence $\|x - y\|_\infty \geq |x(n)| = 1$, which implies c_{00} is not dense in c with respect to $\|\cdot\|_\infty$.

□

— *THE END* —