THE CHINESE UNIVERSITY OF HONG KONG

Department of Mathematics

MATH4010 Functional Analysis 2022-23 Term 1

Homework 9

Deadline: 2022-11-17 Thursday

Notice:

- All the assignments must be submitted before the deadline.
- Each assignment should include your name and student ID number.

1. If P and Q are projections of a Hilbert space H onto closed subspaces E and E', respectively, and $PQ = QP$, then
$$P + Q - PQ$$
is a projection of H onto $E + E'$.

2. Show that U is a self-adjoint unitary operator if and only if $U = 2P - I$ for some orthogonal projection operator P.

   Remark. The question has been modified by adding the assumption that P is orthogonal. Otherwise, an easy counter-example arises in the Hilbert space $\mathbb{C}^2$: Let $P = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$. Then $P^2 = P$. However, $P^* \neq P$ and $2P - I$ is not self-adjoint.

   Recall that for a projection P, $\{$ P is orthogonal $\}$ $\iff$ $\{$ P is self-adjoint $\}$ $\iff$ $\{$ P is normal $\}$.

3. Show that U is a self-adjoint unitary operator on a Hilbert space H if and only if there exist orthogonal closed subspaces E_1, E_2 such that H is the direct sum $E_1 \oplus E_2$ and for every $x = x_1 + x_2$ with $x_1 \in E_1, x_2 \in E_2$,
$$Ux = x_1 - x_2,$$
that is, U is a reflection.

— *THE END* —